Stock market bubble, human capital and growth[*†]

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Abstract

Bubbly episodes are associated with higher training expenditure and growth. Taking into account this empirical evidence, we study the interaction between the stock market, human capital and growth. We consider an economy with infinite-lived agents, in which firms face credit constraints and finance the general training of their workers. This type of training contributes to the worker's general human capital, increasing his productivity and that of the firm. First, we derive the conditions for the existence of the bubble. Second, bubble episodes are associated with higher growth rates. Stock market bubbles have a credit easing effect, they relax the collateral constraint and improve investment in training. This enhances human capital and thus economic growth.

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1 Introduction

Bubbles have occurred many times in the past and have had major impacts on the macro economy. Examples include the 2007 U.S. recession, the 1990 Japanese crisis and the 1929 economic crash. These bubbles featured spectacular booms, followed by dramatic crashes. Our paper is motivated by two elements. First, according to the World Bank in the US, stock prices collapsed and the longest depression began; the average growth rate fell by 1.94% between 2007 and 2009. According to Statista, over the same period, i.e. after the bubble crash, total U.S. expenditures on training staff payroll have dropped from $302.2 billion to $244.4 billion, a decline of 19.12%. This evidence indicates that there may be an interaction between the stock market bubble, training expenditure and growth. Second, to finance the employee training, firms need to some loans. The recourse to these loans declined dramatically during the U.S. bubble crash. According to Federal Reserve Economic Data (Fred), the bank lending to companies recorded a decline of 25.31% between 2008 and 2010. This evidence suggests that loans may facilitate investment in employee training. The objective of this paper is to build a model that highlights the effect of the bubble on growth, through its effect on training (human capital).

We provide a new mechanism explaining that bubbles are associated with episodes of higher growth. It is based on the investment in the human capital of employees. In the presence of a stock market bubble, the collateral value increases and the credit constraint is relaxed. Thus, a firm can finance more general training of its workers. This leads to an improvement in the human capital of workers and therefore to an acceleration of economic growth.

Futagami and Shibata (2000), Grossman and Yanagawa (1993), or King and Ferguson (1993) show that bubbly episodes are associated with lower economic growth, by absorbing a share of over-saving. This is the called crowding-out effect of bubble. These results are contradicted by many recent papers, which show that bubbles have a crowding-in effect. They provide different explanations for the booming of the growth rate in case of bubble. Hashimoto and Im (2016), Hashimoto and Im (2019) use labor market frictions and show that asset bubbles can exist when the employment rate is high, leading to higher economic growth. Olivier (2000) argues that bubble on productive asset can raise the market value of firms, which promotes firm creation, investment and growth. Hirano and Yanagawa (2016), Kunieda and Shibata (2016), Martin and Ventura (2012), focus on financial market imperfections and show that depending on the restrictiveness of the collateral constraint, asset bubble can promote or hinder growth.

Our paper contributes to the previous literature on the positive effects of bubbles on economic growth. A study close to ours is Raurich and Seegmuller (2019), who use an overlapping generations (OLG) model with human capital investment and a financial constraint. They find that depending on the time cost of rearing children, asset bubble can promote growth. If the time cost per child is sufficiently high, households have only a small number of children. Then, the bubble has a crowding-in effect because it is used to provide loans to finance investments in education. Our approach is different from them in three aspects. First, unlike Raurich and Seegmuller's model, we focus on bubbles in the stock market value of the firm, but not in an intrinsically useless asset. Second, in their approach, bubbles are generated by the OLG model, whereas in ours, stock market bubbles are generated by the presence of credit constraints. Third, Raurich and Seegmuller (2019) consider that it is the househould who invests in her own education (human capita) and faces the financial constraint, while in our approach it is the firm that invests in the training of its employees (human capital of employees), and faces the financial constraint.

Our approach builds on the recent paper by Miao et al. (2016) that deals with the relationship between stock market bubble and unemployment. They introduce endogenous credit constraints into a search model of unemployment. They find that because the asset bubble relaxes credit constraints, firms can finance more hiring spending, which leads to lower unemployment. However, this study differs from ours in two respects. First, Miao et al. (2016) consider that firms finance only the hiring of workers, unlike us who consider that the firms finance not only the hiring but also the training of its workers. Second, the study does not consider the possibility of economic growth and does not address the link between asset bubbles and growth.

This paper fills these gaps. We consider an economy with infinitely lived agents, in which firms hire new workers as Miao et al. (2016) do, and finance the general training of their workers. Both types of investments take place in a frictional labor market (Diamond (1982), Mortensen (1982), and Pissarides (1985)). General training contributes to the general human capital of the worker, increasing his productivity and that of the firm. Some countries offer this type of training. For instance, the German apprenticeship system, where the apprenticeship training is largely general. Firms training apprentices have to follow a prescribed curriculum, and apprentices take a rigorous outside exam in their trade at the end of the apprenticeship. Estimates of the net cost of apprenticeship programs to employers in Germany indicate that, in the 1990s, firms bear a significant financial burden associated with these training investments. The net costs of apprenticeship training may be as high as $6,000 per worker (Acemoglu and Pischke (1999)). Another interesting example comes from the recent growth sector of the US, the temporary help industry. The temporary help firms provide a general training to workers who are on short- term contracts, and receive a fraction of the workers' wages as commission. Workers are under no contractual obligation to the temporary help firm after this training program. Most large temporary help firms offer such training to all willing individuals. Although workers taking part in the training programs do not get paid, all the monetary costs of training are borne by the temporary help firms, giving us a clear example of firm-sponsored general training. In our model, we consider that hiring and training investments are financed by internal funds and external debt. To borrow from a lenders, we follow Andolfatto and Gervais (2006) and assume that the firm pledges a fraction of its efficient workers (assets) as collateral. If the firm defaults on its debt, the lenders can seize a fraction of the human capital of the firm's employees. The remaining fraction represents the default costs.

Our results indicate that, first, the existence of a bubble requires that the Lagrangian multiplier associated with the credit constraint and the fraction of workers' human capital is sufficiently high, and the share of firm's expenditures used to finance hiring is sufficiently small . The firm wants to borrow more by using a fraction of its assets as collateral. When this fraction is high, lenders are willing to lend more in the hope that they will be able to recover more if the firm defaults. This encourages the firm to increase its spending on hiring and training. This leads to a raise in the quantity and quality (human capital) of workers, generating a high stock market value for the firm (stock market bubble). The disposable amount for firm's investment increases with a high Lagrangian multiplier associated with the credit constraint, which pushes up the hiring of workers and the financing of their training, facilitating the appearance of the stock market bubble. The share of firm's expenditure used to finance workers' training should be high. In this case, the human capital of workers increases, and thus the value of the firm.

Second, by comparing the BGP with and without bubble, we show that the bubble is productive and has a crowding-in effect on growth and employment, i.e. the level of growth rate and employment are higher when there is a positive bubble. The main mechanism is is that when there is a bubble in the stock market value of the firm, the credit constraint is relaxed. This allows firms to finance more hiring and training of their workers. This enhances employment on the one hand, and the human capital of workers on the other. Since the latter is the engine of growth, then the growth rate is enhanced by bubble. Conversely, when there is no bubble in the stock market value. In this case, the firm cannot borrow more to finance hiring and training spending. This makes employment, economic growth and firm value low.

Third, we study the effect policy training subsidies on economic growth. During the Great Recession in 2009, the governments applied training funds to bring back the labor market and the economic growth from a recession. These funds programmes were typically implemented in combination with other labor market programmes such as wage subsidies, unemployment benefits. A review carried out by Eurofound revealed that in most EU countries public subsidies were the largest form of support for training employed workers, it accounts for about 50% of public support on labor market (see Figure 1)

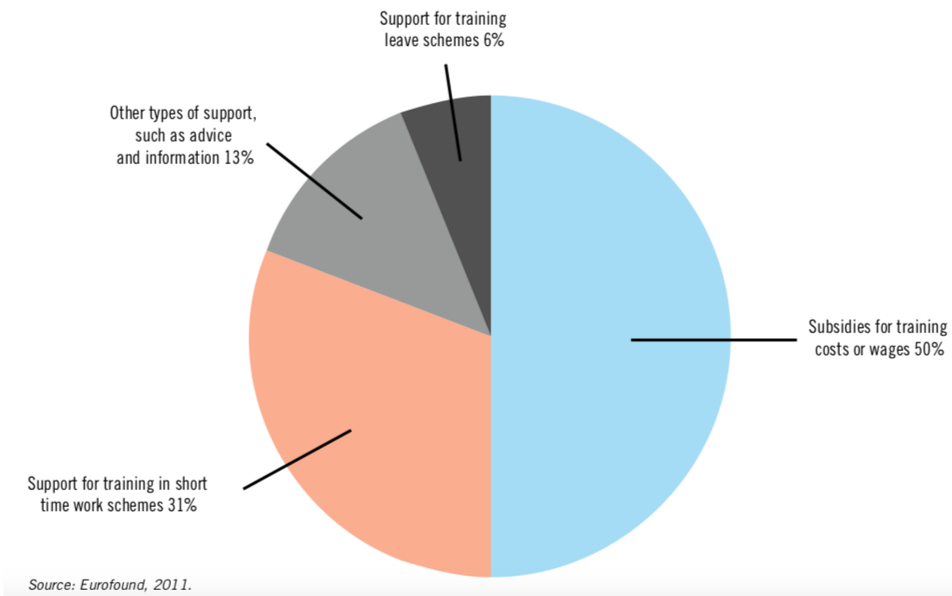

Figure 1 – Public training support schemes for employed workers in EU countries

We find that government subsidies encourage firms to increase their expenditures on employee training. This leads to an increase in the human capital of its worker and thus the growth rate. Although subsidies are intended to finance only the training of workers, we find that they also have a positive effect on the level of employment.

Our paper is organized as follows. The next section presents the model and defines an intertemporal equilibrium. Sections 3 study the bubbleless and bubbly equilibria, respectively. Section 4 analyzes whether a bubble may have a crowding-in effect on growth and employment. Section 5 discusses the effect of training subsidy policy on economic growth. The last section summarizes our findings and concludes the paper. Technical details are relegated to an Appendix

2 Model

We consider a continuous time economy populated by two types of agents, households and firms. We describe the behavior of these two agents and define an intertemporal equilibrium.

2.1 Households

There is a continuum of identical, infinitely-lived households with a unit mass. The representative household derives utility from consumption (C_t) according to the following utility function:

$$\int_0^{\infty} \mathrm{e}^{-\rho t} \log(C_t)\, \mathrm{d}t \tag{1}$$

where $\rho \in [0, 1]$ denotes the subjective discount factor.

In each period of time $t \geq 0$, the representative household is endowed with a unit of time that can be devoted to general training u_t or to work $(1 - u_t)$. The time devoted to training increases the human capital h_t in the next period. Human capital evolves over time through the following accumulation equation which describes the technology of human capital:

$$\dot{h}_t = h_t M_t u_t \tag{2}$$

where M_t represents human capital productivity, which is assumed to depend on firm expenditure general training T_t. General training contributes to the worker's general human capital, increasing his productivity and that of the firm.[1] That is, we define

$$M_t = ET_t \tag{3}$$

where $E > 0$ is a parameter reflecting the efficiency of the training system. The assumption that human capital productivity depends on general training is consistent with many theoretical works (Acemoglu and Pischke (1999), Becker (1964)).

When the representative household works in period t, it receives $W_t = w_t h_t(1 - u_t)$ as labor income, trades the firm's stocks, chooses consumption and the level of its human capital so as to maximize the utility function (1) subject to the budget constraints (2) and:

$$\dot{X}_t = r_t X_t - C_t + N_t w_t h_t (1 - u_t) \tag{4}$$

where X_t, r_t denote wealth and interest rate, respectively. From the household optimization problem, we can immediately derive the following first-order conditions:

$$g_{C_t} = \frac{\dot{C}_t}{C_t} = r_t - \rho \tag{5}$$

and

$$\frac{\dot{N}_t}{N_t} + \frac{\dot{w}_t}{w_t} - \frac{\dot{T}_t}{T_t} = r_t - ET_t \tag{6}$$

[1] Becker (1964) drew a crucial distinction between general versus firm specific training, where the former refers to those skills that are also useful to all or some other employers and the latter concerns the specific skills that increase the productivity of a worker only in the current firm.

2.2 Firms

There is a continuum of heterogeneous firms of a unit measure, owned by households. Each firm $j \in [0,1]$ purchases K_t^j machines at cost k and hires $N_t^j h_t(1-u_t)$ efficiency labor to produce output Y_t^j according to Leontief technology:

$$Y_t^j = A\ min\left\{K_t, [N_t^j h_t(1-u_t)]\right\} \tag{7}$$

To better understanding and facilitate exposition, we sometimes consider a discrete-time approximation in which time is denoted by $t = 0, dt, 2dt$......The continuous time model is the limit when dt goes to zero.

In a frictional labor market and in a small time interval [t, t+dt], each firm j faces two types of investment. The first is the recruitment of new workers $F_t^j h_{t+dt}$ (similar to Miao et al. (2016)), and the second is the financing of the general training for its workers $T_t N_t^j h_t u_t$. Using equations (2) and (3), thus employment in firm j evolves according to:

$$N_{t+dt}^j h_{t+dt} = [(1-s_t dt)N^j + F_t^j dt][1 + ET_t u_t dt]h_t \tag{8}$$

where $s_t > 0$ is the endogenous separation rate. Define aggregate employment as $N_t = \displaystyle\int_0^1 N_t^j dj$ and total hires as $F_t = \displaystyle\int_0^1 F_t^j dj$. Thus, the aggregate employment dynamics becomes:

$$N_{t+dt}h_{t+dt} = \left[(1-s_t dt)N_t + F_t dt\right]\left[1 + ET_t u_t dt\right]h_t \tag{9}$$

In the continuous time, this equation becomes:

$$h_t\dot{N}_t + N_t\dot{h}_t = \big[(-s_t + ET_t u_) N_t + F_t\big] h_t \tag{10}$$

Using equation (2), we reduce the last equation to:

$$\dot{N}_t = -s_t N_t + F_t \tag{11}$$

Define an index of market tightness (job finding rate) as :

$$\theta_t = \frac{F_t}{U_t} \tag{12}$$

where

$$U_t = 1 - N_t \tag{13}$$

denotes the unemployment rate. Assume that the total hiring costs for each firm are given by $G_t F_t^j$, where G_t is an increasing function of market tightness (θ_t):

$$G_t = \sigma \theta_t^{\gamma} \tag{14}$$

where $\sigma > 0$ and $\gamma > 0$ are parameters. Intuitively, if total hires in the market are large relative to unemployment, workers will be relatively scarce and a firm's hiring will be relatively costly.

Suppose that the hiring new workers and investment in general training are financed by internal funds $\left[A - w_t\right] N_t^j h_t(1-u_t)$ and external debt L_t^j as follow:

$$[k(1-u_{t+dt}) + G_t] F_t^j h_{t+dt} + T_t N_t^j h_t u_t \leqslant \left[A - w_t\right] N_t^j h_t(1-u_t) + L_t^j \tag{15}$$

where the external debt (collateral constraint) is given by

$$L_t^j \leqslant \mathrm{e}^{-\rho dt}\, V_{t+dt}\big(\xi N_t^j h_t(1-u_t)\big) \tag{16}$$

where $\xi \in (0,1)$, and $V_t(N_t^j h_t(1-u_t))$ is the market value of the firm j at time t. Consider that each firm j faces the same degree of pledgeability, represented by the parameter ξ. This parameter represents the degree of financial frictions. In order to borrow, firm j pledges a fraction of its assets (capital stock K_t^j) as collateral. Due to the Leontief technology, it pledges efficient labor $N_t^j h_t(1-u_t)$ as collateral.[2] If the firm defaults on its debt, lenders can capture $\xi N_t^j h_t(1-u_t)$ assets of the firm. The remaining fraction $(1-\xi)$ accounts for default costs. The Lenders and the firm renegotiate the debt and the lenders keep the firm running in the next period $t+dt$. Thus lenders can get the threat value $\mathrm{e}^{-rdt}\, V_{t+dt}(\xi N_t^j h_t(1-u_t))$. In the continuous-time limit as $dt \to (0,1)$ becomes:

$$L_t^j \leqslant V_t\big(\xi N_t^j h_t(1-u_t)\big) \tag{17}$$

[2] Educated labor is more complementary with physical capital than uneducated labor (Griliches (1969).

One combine the financial (15) and collateral constraints (16) and obtain:

$$[k(1-u_{t+dt})+G_t]F_t^j h_{t+dt}+T_t N_t^j h_t u_t \leqslant \big[A-w_t\big]N_t^j h_t(1-u_t)+ e^{-\rho dt}V_{t+dt}\big(\xi N_t^j h_t(1-u_t)\big) \quad (18)$$

The market value of the firm j $V_t(N_t^j h_t(1-u_t))$ satisfies the following Bellman equation in the discrete-time approximation:

$$V_t(N_t^j h_t(1-u_t)) = \max_{F_t^j, T_t} \quad \big[A-w_t\big]N_t^j h_t(1-u_t)dt - [k(1-u_{t+dt})+G_t]F_t^j h_{t+dt}dt - T_t N_t^j h_t u_t dt + \mathrm{e}^{-\rho dt}\, V_{t+dt}\bigg([(1-sdt)N_t^j h_t + F_t^j h_t](1+ET_t u_t dt)(1-u_{t+dt})\bigg) \quad (19)$$

where we consider that the firm value j takes the following form:

$$V_t\big(N_t^j h_t(1-u_t)\big) = Q_t N_t^j h_t(1-u_t) + B_t^j \quad (20)$$

Q_t and $B_t^j \equiv N_t^j h_t b_t$ are variables to be determined. We may interpret $Q_t N_t^j h_t(1-u_t)j$ as the fundamental value of the firm. $b_t > 0$ is the value of the bubble per efficiency of labor (the normalized bubble).

Each firm j maximizes the problem (19) subject to the financial constraint (18). The first-order condition with respect to F_t^j and T_t, respectively is:

$$\big[k(1-u_t)+G_t\big](1+\mu) = (1-u_t)Q_t + b_t \quad (21)$$

$$(1+\mu) = E[(1-u_t)Q_t + b_t] \quad (22)$$

where $\mu > 0$ denotes the Lagrangian multiplier associated with the credit constraint (18). If $\mu = 0$, then the model reduces to the case with perfect capital markets.

2.3 Nash bargaining

We derive the equilibrium wage rate. In frictional labor market, the wage is determined by Nash bargaining process as in DMP (Diamond-Mortensen-Pissarides) model with human capital as follow:

$$\max_{w_t}(S_t^H)^{\nu}(S_t^F)^{(1-\nu)} \tag{23}$$

where S_t^H and S_t^F are the household and firm surplus, respectively. The values of an employed and unemployed satisfy the following equations:

$$V_t^N = w_t h_t(1-u_t)dt + \mathrm{e}^{-\rho dt}[s_t V_{t+dt}^U dt + (1-s_t dt)V_{t+dt}^N] \tag{24}$$

$$V_t^U = \mathrm{e}^{-\rho dt}[\theta_t V_{t+dt}^N dt + (1-\theta_t dt)V_{t+dt}^U] \tag{25}$$

Thus, the household surplus is given by:

$$\begin{aligned} S_t^H &= V_t^N - V_t^U \\ &= w_t h_t(1-u_t)dt + \mathrm{e}^{-\rho dt}(1-s_t dt-\theta_t dt)S_{t+dt}^H \end{aligned} \tag{26}$$

Suppose that $\mu > 0$, the firm surplus S_t^F is derived by applying the envelop theorem to problem (19):

$$\begin{aligned}
S_t^F = \frac{\partial V_t\big(N_t^j h_t(1-u_t)\big)}{\partial N_t^j} &= (1+\mu)\big(A-w_t)h_t(1-u_t)dt - (1+\mu)T_t h_t u_t dt + \\
&\quad \mathrm{e}^{-\rho dt}\, h_{t+dt}(1-s_t dt)[(1-u_{t+dt})Q_{t+dt} + b_{t+dt}] + \\
&\quad \mathrm{e}^{-\rho dt}\, \mu dt\big[\xi h_t(1-u_t)Q_{t+dt} + h_{t+dt}(1-s_t dt)b_{t+dt}\big] \\
&= (1+\mu)\big(A-w_t)h_t(1-u_t)dt - (1+\mu)T_t h_t u_t dt + \\
&\quad \mathrm{e}^{-\rho dt}(1-s_t dt)S_{t+dt}^F + \mathrm{e}^{-\rho dt}\, \mu dt\Big[\xi h_t(1-u_t)Q_{t+dt} + \\
&\quad h_{t+dt}(1-s_t dt)b_{t+dt}\Big] \qquad (27)
\end{aligned}$$

Using equations (23), (26) and (27), the Nash bargained wage (w_t) solves the following problem:

$$\begin{aligned}
&\max_{w_t}\Big[w_t h_t(1-u_t)dt + \mathrm{e}^{-\rho dt}(1-s_t dt-\theta_t dt)S_{t+dt}^H)\Big]^{\nu} \times \\
&\quad\Big[(1+\mu)\big(A-w_t)h_t(1-u_t)dt - (1+\mu)T_t h_t u_t dt + \mathrm{e}^{-\rho dt}(1-s_t dt)S_{t+dt}^F \\
&\qquad + \mathrm{e}^{-\rho dt}\, \mu dt\big[\xi h_t(1-u_t)Q_{t+dt} + h_{t+dt}(1-s_t dt)b_{t+dt}\big]\Big]^{(1-\nu)} \qquad (28)
\end{aligned}$$

where $0 < \nu < 1$ denotes the relative bargaining power of the worker. The first order condition implies that

$$\nu S_t^F = (1-\nu)(1+\mu)S_t^H \qquad (29)$$

Note that this sharing rule is different from the standard Nash bargaining solution in the DMP model. The presence of credit constraints induces a wedge given by $\mu > 0$. An additional dollar increase in the wage rate raises the value of the employed by one dollar. But it decreases firm value by $(1+\mu)$ dollars, which is more than one dollar. The additional cost μ to the firm reflects costly external finance caused by credit constraints.

Since we have assumed that wage is negotiated continually, the last equation holds in rates of change as in Pissarides (2000), we thus obtain:

$$\nu \dot{S_t^F} = (1-\nu)(1+\mu)\dot{S_t^H} \tag{30}$$

The Nash bargained wage is derived from equation (30) (see Appendix A.1)

$$w_t = \nu \left[A - \frac{T_t u_t}{(1-u_t)} + \frac{\xi\mu + \theta_t}{(1+\mu)} Q_t + \frac{\mu + \theta_t}{(1+\mu)(1-u_t)} b_t \right] \tag{31}$$

This equation shows that at time $t > 0$, the wage per efficient labor (w_t) is increasing in A, ξ,θ_t, Q_t, b_t, and decreasing in T_t. A higher value of the job finding rate θ_t, implies that a searcher can more easily find a job and hence demand a higher wage. Workers get higher wages when the marginal Q_t of the firm, the bubble b_t and the share ξ are high. The effect of learning time u_t on w_t depends on the bubble value. When $b_t = 0$, a higher u_t decreases the wage rate. When $b_t > 0$, w_t always decreases with u_t if $(1+\mu)T_t > (\mu + \theta_t)b_t$ otherwise it increases. However, at time $t + dt > 0$, learning time u_t and training T_t affect positively the wage per labor W_{t+dt} because in the next period, each worker gets $W_{t+dt} = w_{t+dt}(1 + ET_t u_t)(1 - u_{t+dt})h_t$. Thus, the workers accept to be trained because they perceive a higher wage in the next period. Regarding the effect of the Lagrangien multiplier associated to the financial constraint μ on the wage w_t, we rewrite the wage rate by using equation (22) as follows:

$$w_t = \nu\left[A - \frac{T_t u_t}{(1-u_t)} + \frac{\xi\mu + \theta_t}{E(1-u_t)} + \frac{\mu(1-\xi)}{(1+\mu)(1-u_t)}b_t\right] \tag{32}$$

we deduce that a higher μ leads to a higher wage rate. In competitive labor markets where workers receive a wage W_t equal to their marginal product $Ah_t(1-u_t)$, firms cannot recoup investments in general skills, which implies that they refuse to pay for general training. This is because other firms may benefit from this investment in the event that workers separate from the firm and work for another firms. Each firm finances the general training of its workers only in frictional labor market (Acemoglu and Pischke (1999)). The worker then obtains a wage that is below his marginal product (MPL). This gap between MPL and market wage is not sufficient to ensure firm-sponsored investments in general training. So that this happens, the workers' wage must be compressed; i.e. the marginal product of labor that increases with skills (MPL') is higher than the wage that increases with skills (W_t'). In our model, the wage is compressed if:

$$\begin{aligned} W_t' &< MPL' \Rightarrow \\ \nu\left[A(1-u_t) - T_t + \frac{\xi\mu + \theta_t}{(1+\mu)}Q_t(1-u_t) + \frac{\mu + \theta_t}{(1+\mu)}b_t\right] &< A(1-u_t) \end{aligned} \tag{33}$$

Suppose that the wage is compressed; i.e. condition (33) is satisfied. In this case, the firm makes more profits from a more skilled (trained) worker, and has an incentive to increase the skills of the worker by investing in the general skills of its employees.

2.4 Equilibrium dynamics

In this subsection, we characterize the equilibrium dynamics of the economy. We conduct aggregation where aggregate employment is $N_t = \int_0^1 N_t^j dj$ and total hires is $F_t = \int_0^1 F_t^j dj$. Since the conjecture of the value function of the firm takes the form (20), then the continuous-time limit of the Bellman equation (19) is given by (see Appendix A.2):

$$\begin{aligned} N_t(1-u_t)\dot{Q}_t + N_t\dot{b}_t &= \big[(\rho + s_t - g_{h_t})N_t - F_t\big]\big[(1-u_t)Q_t + b_t\big] \\ &\quad -[A - w_t](1-u_t)N_t + [k + G_t]F_t \\ &\quad +T_tN_t \pm \gamma N_t Q_t \end{aligned} \tag{34}$$

Applying the envelop theorem to equation (34), the differential equation of Q_t is given by:

$$\dot{Q}_t = \big[\rho + s_t - g_{h_t} - \frac{F_t}{N_t} - \xi\mu\big]Q_t - (1+\mu)\big[A - w_t\big] \tag{35}$$

We can interpret Q_t as the shadow price of capital K_t^j (or efficient of labor $N_t^j h_t(1-u_t)$). To extract the value of the interest rate from (35), we use equations (5) and (11):

$$r_tQ_t = \dot{Q}_t + [g_{h_t} + g_{N_t} + g_{c_t} + \xi\mu]Q_t + (1+\mu)(A - w_t) \tag{36}$$

where $g_{h_t} = \frac{\dot{h}_t}{h_t}$, $g_{N_t} = \frac{\dot{N}_t}{N_t}$ and $g_{C_t} = \frac{\dot{C}_t}{C_t}$ denote the growth rate of human capital, the growth rate of employment and the growth rate of consumption, respectively. The last equations says that the return on capital $r_t Q_t$ is equal to capital gains $\dot{Q}_t$ plus dividends $[g_{h_t} + g_{N_t} + g_{c_t} + \xi\mu]Q_t + (1+\mu)(A - w_t)$.

The differential equation of b_t is obtained by using equations (34) and (35):

$$\dot{b}_t = \big[1 + \rho + s_t - g_{h_t} - \frac{F_t}{N_t}]b_t + (1+\mu)\big[(A - w_t)(1 - u_t) + \xi Q_t(1 - u_t)\big] \quad (37)$$

To interpret the asset-pricing equation for the bubble, using equation (5) and (11) we rewrite (37) as:

$$r_t = \frac{\dot{b}_t}{b_t} + g_{h_t} + g_{N_t} + \left[g_{c_t} - \frac{(1+\mu)}{b_t}\big[(A - w_t)(1 - u_t) + \xi Q_t(1 - u_t) + b_t\big]\right] \quad (38)$$

According to this equation, the rate of return on the firm's bubble is equal to the interest rate r_t. This return consists of two components: the first is the rate of capital gains represented by $\frac{\dot{b}_t}{b_t} + g_{h_t} + g_{N_t}$, and the second is the dividend yield $g_{c_t} - \frac{(1+\mu)}{b_t}\big[(A - w_t)(1 - u_t) + \xi Q_t(1 - u_t) + b_t\big]$. Each unit of the bubble raises the collateral value by one unit and hence allows the firm to borrow an additional unit to finance hiring and training cost. This unit of investment raises firm value by g_{c_t}, Subtracting costs of hiring workers and investment into capital and training $\frac{(1+\mu)}{b_t}\big[(A - w_t)(1 - u_t) + \xi Q_t(1 - u_t) + b_t\big]$. We deduce that the second term on the right-hand side of (38) represents the net increase in the firm's value for each unit of bubbles. This is why we call this term dividend yields.

Definition 1 *Suppose $\mu > 0$. Then the equilibrium for this economy consists of the consequences (g_{c_t}, r_t, T_t, w_t, u_t N_t, F_t, U_t, θ_t, G_t, Q_t, b_t, s_t, g_{h_t}), that satisfy the following conditions:*

a) Household utility maximization: equations (5) and (6)

b) Firm maximization: equations (21) and (22)

c) Nash bargaining solution: equation (31)

and satisfy also the system of equations (2), (11), (12), (13), (14), (35), (37), the optimal hiring and training investment, respectively

$$F_t = \beta \frac{(A - w_t)N_t(1 - u_t) + \xi N_t(1 - u_t)Q_t + N_t b_t}{k(1 - u_t) + G_t} \tag{39}$$

$$g_{h_t} = (1 - \beta)\Big[(A - w_t)(1 - u_t) + \xi(1 - u_t)Q_t + b_t\Big] \tag{40}$$

where β $(1-\beta)$ is the share of the financing used to finance hiring workers (training)

There are clearly two types of equilibria. The first type is bubbleless, for which $b_t = 0$ for all t: In this case, the market value of firm j is equal to its fundamental value; i.e. $V_t(N_t^j h_t(1 - u_t)) = Q_t N_t^j h_t(1 - u_t)$. The second type is bubbly, for which $b_t > 0$ for some t. In this case, the firm's value contains a bubble component in that $V_t\big(N_t^j h_t(1 - u_t)\big) = Q_t N_t^j h_t(1 - u_t) + B_t^j$. In the next section, we studies these two types of equilibria in balanced growth path (BGP).

3 Balanced growth path with and without bubble

A balanced growth path BGP (or steady state) is a solution where the paths (h_t, c_t) grow at a constant rate g, and (s_t, r_t, T_t, u_t, Q_t, N_t, U_t, θ_t, F_t, G_t, w_t, b_t) remain constant for all $t > 0$, satisfying definition 1 (see Appendix A.3).

There are obviously two types of stationary equilibria, the bubbleless ones with $b = 0$ and the bubbly ones with $b > 0$. We first study the existence and uniqueness of a bubbleless BGP. Solving the above system in BGP with $b = 0$ yields:

Proposition 1 *There exists a unique bubbleless BGP given by (s, g, r, T, u, Q, G, θ, N, U, F, w, b)=($\tilde{s}$, $\tilde{g}$, $\tilde{r}$, $\tilde{T}$, $\tilde{u}$, $\tilde{Q}$, $\tilde{G}$, $\tilde{\theta}$, $\tilde{N}$, $\tilde{U}$, $\tilde{F}$, $\tilde{w}$, 0)*

where their values take the following form (see appendix (A.3))

$$\tilde{g} = \frac{1-\beta}{2-\beta}(\rho+\xi) \qquad \tilde{G} = \frac{1}{E} - k(1-\tilde{u})$$

$$\tilde{s} = \frac{\beta}{1-\beta}\tilde{g} \qquad \tilde{\theta} = [\frac{\tilde{G}}{\sigma}]^{\frac{1}{\gamma}}$$

$$\tilde{r} = \tilde{g} + \rho \qquad \tilde{N} = \frac{\tilde{\theta}}{\tilde{\theta}+\tilde{s}}$$

$$\tilde{T} = \frac{\tilde{r}}{E} \qquad \tilde{U} = 1 - \tilde{N}$$

$$\tilde{u} = \frac{\tilde{g}}{E\tilde{T}} \qquad \tilde{F} = \tilde{s}\tilde{N}$$

$$\tilde{Q} = \frac{1+\mu}{E(1-\tilde{u})} \qquad \tilde{w} = \nu\left[A - \frac{\tilde{T}}{(1-\tilde{u})} + \frac{\xi\mu+\tilde{\theta}}{(1+\mu)}\tilde{Q}\right]$$

We switch now to the bubbly BGP indexed by a star * where $b^* > 0$.

Proposition 2 *There exists a bubbly BGP with $b^* > 0$ if and only if:*

$$\mu > \underline{\mu} \tag{41}$$

where $\underline{\mu} \equiv \frac{1-\xi}{1-\beta} - \xi$. The bubbly BGP values with $b > 0$ are given by (see Appendix A.4):

$$g^* = \frac{(1-\beta)(\xi+\mu)}{(1-\beta)(\xi+\mu)-(1-\xi)}(\rho+\xi) \qquad G^* = \frac{1}{E} - k(1-u^*)$$

$$s^* = \frac{\beta}{1-\beta}g^* \qquad \theta^* = [\frac{G^*}{\sigma}]^{\frac{1}{\gamma}}$$

$$r^* = g^* + \rho \qquad N^* = \frac{\theta^*}{\theta^* + s^*}$$

$$T^* = \frac{r^*}{E} \qquad U^* = 1 - N^*$$

$$u^* = \frac{g^*}{ET^*} \qquad F^* = s^* N^*$$

$$b = \frac{(1+\mu)[g^* - (\rho+\xi)]}{E(1-\xi)} \qquad w^* = \nu\left[A - \frac{T^* u^*}{(1-u^*)} + \frac{\xi\mu + \theta^*}{(1+\mu)}Q^* + \frac{\mu+\theta^*}{(1+\mu)(1-u^*)}b\right]$$

$$Q^* = \frac{1+\mu}{E(1-u^*)} - \frac{b}{E(1-u^*)}$$

Inequality (41) ensures that $b^* > 0$. In this case, the bubbly BGP coexists with bubbleless BGP. By inspection this inequality, we see that Proposition 2 will be satisfied if the Lagrangian multiplier associated with the credit constraint μ and the fraction of workers' human capital ξ are sufficiently high, and the share of firm's expenditures used to finance hiring β is sufficiently low. The firm wants to borrow more using a fraction ξ of its assets as collateral. When this fraction ξ is high, lenders are willing to lend more in the hope that they will be able to recover more if the firm defaults. This encourages firm to increase its spending on hiring and training. This leads to a raise in the quantity and quality (human capital) of workers, generating a high stock market value of the firm (stock market bubble). The disposable amount for firm's investment increases with μ, which pushes up the hiring of workers and the funding of their training, facilitating the appearance of the stock market bubble. High $(1-\beta)$, means that the share of firm's expenditure used to finance workers' training should be high. In this case, the human capital of the workers increases, and therefore the firm value too.

When the model is without credit constraints; i.e. $\mu = 0$ and $\xi = 0$, the capital markets are perfect. In this case, inequality (41) is not satisfied. Thus, when households and workers have infinite lives and credit markets are perfect, the bubble cannot exist. This implies that the presence of the credit constraint is paramount for the existence of stock market bubble.

4 Human capital and the crowding-in effect of bubbles

We now examine whether the existence of asset bubble raises employment and growth. By comparing the bubbly and bubbleless BGP, we will be able to deduce whether the bubbly steady state is characterized by higher levels of growth and employment.

Let us start by examining the effect of bubble on employment. Hiring and unemployment BGP equilibrium are determined through the interaction between the two equations $F = s(1-U)$ and $F = \theta U$. When the bubble exists; i.e. $\mu > \underline{\mu}$, the separation rate increases from $\tilde{s}$ to s^* and the job finding raises from $\tilde{\theta}$ to θ^*. This generates a raise of hiring worker from $\tilde{F}$ to F^* and a decline of unemployment from $\tilde{U}$ to U^*. This in turn implies an upward shift in employment from $\tilde{N}$ to N^*. These results are illustrated in Figure 2, where the blue curve represents the Beveridge curve.

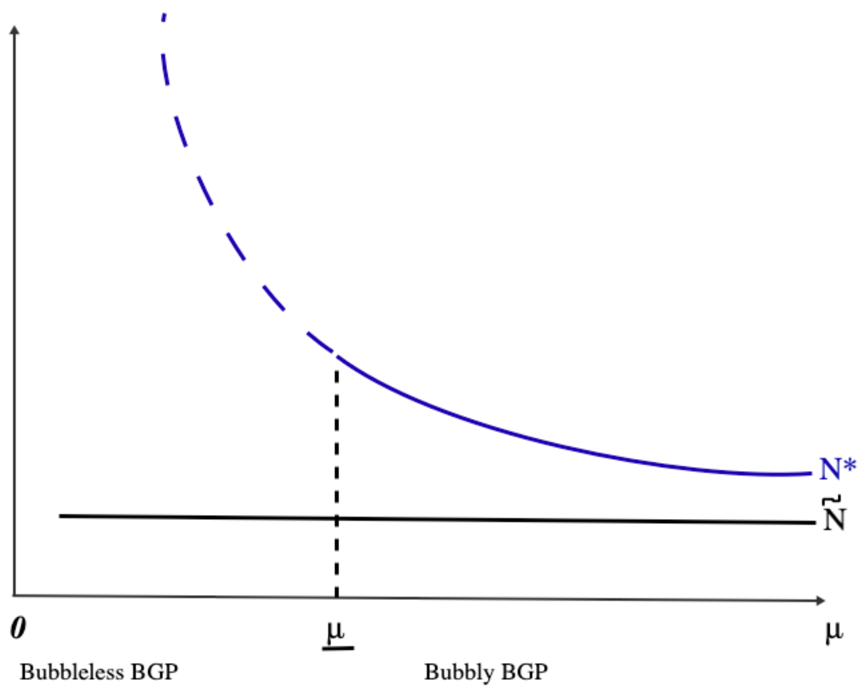

Figure 2 – Hiring and unemployment in bubbly and bubbleless equilibrium

In Miao and Wang model, the separation rate is exogenous and the job finding rate (market tightness) is endogenous. They find that when there is a bubble, the job finding rate is higher and the separation rate remains constant. In this case, only the dark curve moves upward. Unlike our model, the separation rate is endogenous. Thus, we find that the finding and separation rates are higher in bubbly BGP. In this case, both the dark and blue curves move upwards. This generates a high employment rate in bubbly BGP; i.e. $N^* > \tilde{N}$.

How does the stock market bubble affect the growth rate? To answer this question, we compare the growth rate with and without bubble, we easily find that $g^* > \tilde{g}$. The results are summarized in the following proposition:

Proposition 3 *Assume that inequality (41) holds.*

1. *Employment is higher at the bubbly BGP, i.e.* $N^* > \tilde{N}$

2. *Growth rate is higher at the bubbly BGP, i.e.*$g^* > \tilde{g}$.

The presence of an asset bubble allows firms to access the credit constraint more easily, allowing them to borrow more to finance the hiring and training ($T^* > \tilde{T}$) of their workers. This generates an increase in employment on the one side, and in human capital on the other. Since human capital is an important factor contributing to long-run growth, this results in a higher growth rate when the bubble exists. Therefore, we show that thanks to human capital, asset bubble is productive and has a crowding-in effect on growth. Conversely, when there is no stock market bubble, the credit constraint is tight causing firms to reduce hiring and training workers. This makes employment and economic growth low. The relationship between employment with and without bubble, and the relationship between growth rate with and without bubble are illustrated in Figure 3 and Figure 4, respectively.

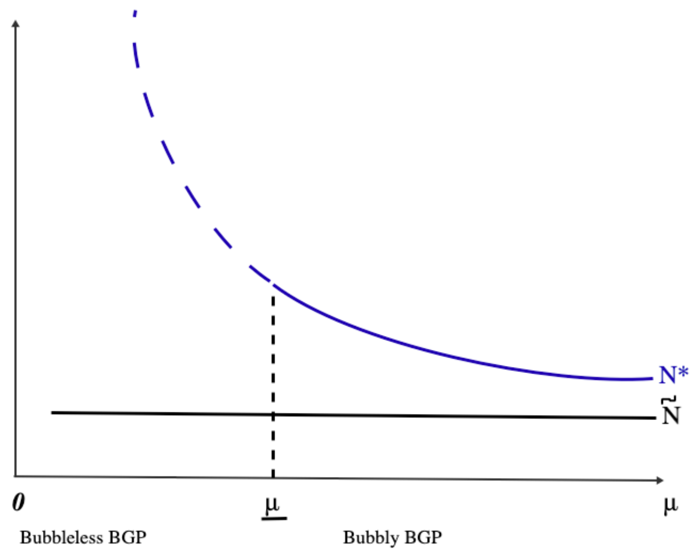

Figure 3 – Employment with and without bubble

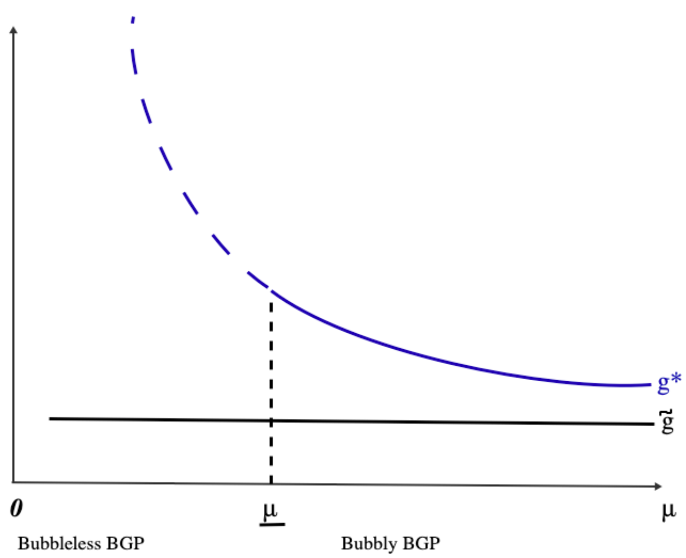

Figure 4 – Growth rate with and without bubble

In addition, following an increase of training in bubbly BGP, each worker spends more time on training, i.e. $u^* > \tilde{u}$. The cost of recruiting skilled (trained) workers becomes higher; i.e. $G^* > \tilde{G}$ as well as the interest rate. This last result is a common finding in the literature about the positive correlation between the bubble and the interest rate.

5 Training subsidies policy

During the Great Recession in 2009, governments implemented several policies to bring back the labor market and economic growth after the recession, such as spending on labor market programmes. It constituted an important part of public spending: wage subsidies, unemployment benefits as well as funds training. Training and retraining programmes were typically implemented in combination with other labor market programmes. This program resulted in more cost-effective support for groups most seriously affected by the crisis, including low-skilled workers. A review carried out by Eurofound revealed that in most EU countries public subsidies were the largest form of support for training employed workers, it accounts for about 50% of public support on labor market.

To investigate the effect of training subsidies policy on human capital as well as on the growth rate, we consider that the government subsidizes the training of each worker by Z value. We keep the previous model as is and introduce the training subsidy policy in the financial constraint so that:

$$\begin{aligned} [k(1-u_t)+G_t]F_t h_t + T_t N_t h_t u_t \;&\leqslant\; \big[A-w_t\big] N_t h_t (1-u_t) + Z N_t h_t \\ &\quad +\xi N_t h_t (1-u_t) Q_t + N_t h_t b_t \end{aligned} \qquad (42)$$

As the subvention concerns only the training, the optimal training investment changes is:

$$g_{h_t} = (1-\beta)\Big[(A-w_t)(1-u_t)+\xi(1-u_t)Q_t+b_t\Big]+Z \tag{43}$$

We follow the same steps as before to solve the model in BGP. We find that the separation rate in bubbleless BGP $\hat{s}$ does not depend on Z; i.e. $\hat{s}=\tilde{s}=\frac{\beta}{2-\beta}(\rho+\xi)$, but the others values that are indexed by $\hat{}$ take the following form:

$$\hat{g} = \tilde{g} + Z \qquad \hat{G} = \frac{1}{E} - k(1-\hat{u})$$

$$\hat{r} = \hat{g} + \rho \qquad \hat{\theta} = [\frac{\hat{G}}{\sigma}]^{\frac{1}{\gamma}}$$

$$\hat{T} = \frac{\hat{r}}{E} \qquad \hat{N} = \frac{\hat{\theta}}{\hat{\theta}+\tilde{s}}$$

$$\hat{u} = \frac{\hat{g}}{E\tilde{T}} \qquad \hat{U} = 1 - \hat{N}$$

$$\hat{Q} = \frac{1+\mu}{E(1-\hat{u})} \qquad \hat{F} = \tilde{s}\hat{N}$$

$$\hat{w} = \nu\Big[A - \frac{\hat{T}}{(1-\hat{u})} + \frac{\xi\mu+\hat{\theta}}{(1+\mu)}\hat{Q}\Big]$$

We deduce easily that training subsidies Z have a positive impact on the growth rate g. This is because a high Z allows the firm to raise its investment in training T, which leads to an increase in the human capital of its worker and thus in the growth rate.

Although the subsidies are intended to finance only the training of workers, they also have an effect on the level of employment. The upward trend in Z leads to an increase in the job finding rate, while the separation rate remains constant. This leads to an increase in the hiring of workers and a decrease in the unemployment rate. Graphically, a high Z pushes the $H = \theta U$ curve upwards, while the $H = s(1 - U)$ curve remains constant. Figure 5 illustrates the result.

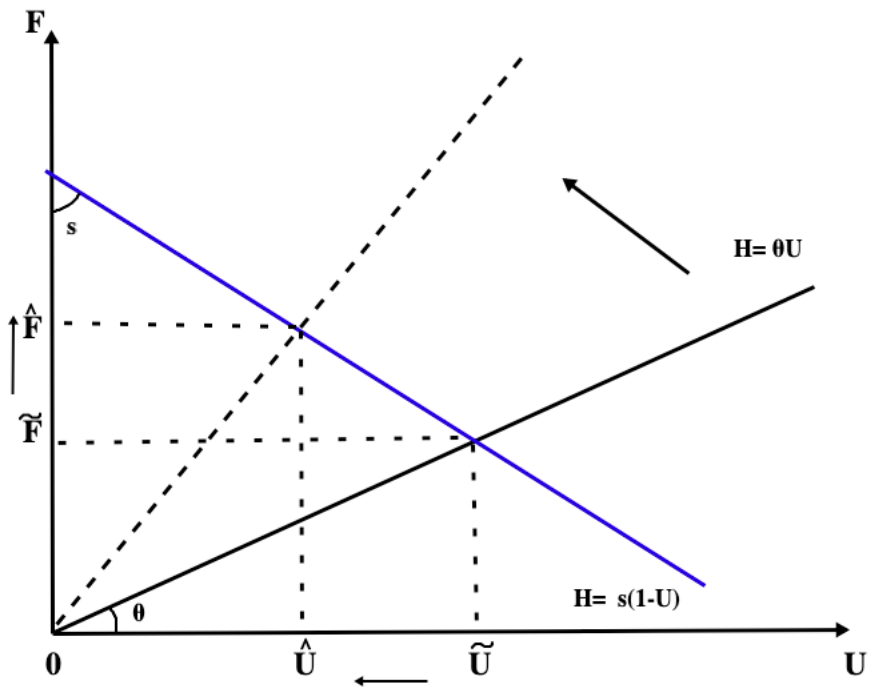

Figure 5 – Hiring and unemployment in bubbleless equilibrium when training subsidies increases

Higher human capital is associated with higher training subsidies, which leads to higher firm productivity. This encourages firms to hire more. Therefore, we show that through training subsidies, the growth rate and employment increase.

6 Conclusion

This paper has highlighted the effect of asset bubble on growth through human capital. We thus considered an economy with infinitely lived agents, in which firms hire new workers as Miao et al. (2016), and finance general training of their workers in frictional labor market. Both investments are financed by internal funds and loan. To borrow, each firm pledges a fraction of its efficient workers as collateral. We have shown that the existence of bubble requires i) high level of the Lagrangian multiplier associated with the credit constraint, ii) high fraction of workers' human capital and iii) low share of firm's expenditures used to finance hiring. The comparison between equilibria with and without bubble showed that the growth rate and employment are higher when there is a bubble. The latter relaxes the credit constraints and allows firms to make more investment in hiring and training workers. This generates an upward movement in employment and growth. Conversely, when the is no bubble in the economy, the credit constraint tightens. As a result, employment, human capital and economic growth are low.

In terms of policy implications, we have shown that training funds allow firms to increase their investment in the human capital of their workers, which implies higher economic growth and employment.

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A Appendix

A.1 Nash bargaining wage

We begin by determining the value of $\dot{S_t^F}$. We introduce S_{t+dt}^F in equation (27), and we get:

$$S_{t+dt}^F - S_t^F = -(1+\mu)\big(A - w_t\big)h_t(1-u_t)dt + (1+\mu)T_t h_t u_t dt + \\ S_{t+dt}^F - \mathrm{e}^{-\rho dt}(1 - s_t dt)S_{t+dt}^F - \mathrm{e}^{-\rho dt}\,\mu dt\Big[\xi h_t(1-u_t)Q_{t+dt} + h_{t+dt}(1-s_t dt)b_{t+dt}\Big] \tag{44}$$

To compute the limit, we use the heuristic rule $dE_t = E_{t+dt} - E_t$ for any variable E_t, we also use the notation $\dot{E}_t = \frac{dEt}{dt}$. Notice that $\lim\limits_{dt\to 0} \frac{\mathrm{e}^{-\rho dt}-1}{dt} = 1$. So when dt $\to$ 0, then:

$$\dot{S_t^F} = -(1+\mu)\big(A - w_t\big)(1-u_t)h_t + (1+\mu)T_t h_t u_t + [\rho + s_t - \mu\xi]S_t^F - \mu(1-\xi)b_t h_t \tag{45}$$

We switch to determine $\dot{S_t^H}$. We introduce S_{t+dt}^H in equation (26), we get:

$$S_{t+dt}^H - S_t^H = -w_t h_t(1-u_t)dt + S_{t+dt}^H - \mathrm{e}^{-\rho dt}(1 - s_t dt - \theta_t dt)S_{t+dt}^H \tag{46}$$

So when dt $\to 0$, then:

$$\dot{S_t^H} \quad = \quad -w_t h_t(1-u_t) + (\rho + s_t + \theta_t) S_t^H \tag{47}$$

where the value of S_t^H is obtained from (29)

$$S_t^H = \frac{\nu[(1-u_t)h_t Q_t + h_t b_t]}{(1-\nu)(1+\mu)} \tag{48}$$

Therefore the $\dot{S_t^H}$ becomes:

$$\dot{S_t^H} \quad = \quad -w_t h_t(1-u_t) + (\rho + s_t + \theta_t)\frac{\nu[(1-u_t)h_t Q_t + h_t b_t]}{(1-\nu)(1+\mu)} \tag{49}$$

A.2 Continous-time limit of Bellman equation

We know that both equations (19) and (20) are equal; i.e.

$$\begin{aligned} Q_t N_t h_t(1-u_t) + N_t h_t b_t = \big[A - w_t\big] N_t h_t (1-u_t) dt - [k(1-u_{t+dt}) + G_t] F_t h_{t+dt} dt \\ - T_t N_t h_t u_t dt + \mathrm{e}^{-\rho dt}\Bigg[[(1-sdt)N_t h_t + F_t h_t](1 + ET_t u_t dt)\Bigg]\big[(1-u_{t+dt})Q_{t+1} + b_{t+1}\big] \end{aligned} \tag{50}$$

We introduce $V_{t+dt}\big(N_{t+dt}h_{t+dt}(1-u_{t+dt})\big)$ in both sides of the equality:

$$\begin{aligned} & Q_{t+dt}N_{t+dt}h_{t+dt}(1-u_{t+dt}) + N_{t+dt}h_{t+dt}b_{t+dt} - Q_tN_th_t(1-u_t) - N_th_tb_t = \\ & \quad Q_{t+dt}N_{t+dt}h_{t+dt}(1-u_{t+dt}) + N_{t+dt}h_{t+dt}b_{t+dt} - \big[A-w_t\big]N_th_t(1-u_t)dt \\ & \quad + [k(1-u_{t+dt}) + G_t]F_th_{t+dt}dt + T_tN_th_tu_tdt \\ & \quad - \mathrm{e}^{-\rho dt}\Big[[(1-sdt)N_th_t + F_th_t](1+ET_tu_tdt)\Big]\big[(1-u_{t+dt})Q_{t+1} + b_{t+1}\big] \end{aligned} \tag{51}$$

We suppose that $u_{t+dt} = u_t \pm \gamma dt$, where γ is a parameter. We substitute equation (9) in (51), to get:

$$\begin{aligned} & N_t(1-u_t)(Q_{t+dt}-Q_t) + N_t(b_{t+dt}-b_t) + dt\big[ET_tN_tu_t + \\ & \quad (F_t - s_tN_t)(1+ET_tu_tdt)\big]\big[(1-u_{t+dt})Q_{t+dt} + b_{t+dt}\big] \pm \gamma h_tN_tQ_{t+dt}dt = \\ & \quad - \big[A-w_t\big]N_th_t(1-u_t)dt + [k(1-u_{t+dt}) + G_t]F_th_{t+dt}dt + T_tN_th_tu_tdt \\ & \quad + (1-\mathrm{e}^{-\rho dt})N_{t+dt}h_{t+dt}\big[(1-u_{t+dt})Q_{t+dt} + b_{t+dt}\big] \end{aligned} \tag{52}$$

When $dt \to 0$ the Belleman equation becomes:

$$\begin{aligned} N_t(1-u_t)\dot{Q}_t + N_t\dot{b_t} &= \big[(\rho + s_t - g_{h_t})N_t - F_t\big]\big[(1-u_t)Q_t + b_t\big] \\ & \quad -[A-w_t](1-u_t)N_t + [k+G_t]F_t \\ & \quad +T_tN_t \pm \gamma N_tQ_t \end{aligned} \tag{53}$$

A.3 Bubbleless BGP solutions

The BGP system is given by the following equations:

$$g = ETu \tag{54}$$

$$\rho = r - g \tag{55}$$

$$r = ET \tag{56}$$

$$s = \frac{F}{N} \tag{57}$$

$$U = 1 - N \tag{58}$$

$$\theta = \frac{F}{U} \tag{59}$$

$$G = \mu\theta^{\gamma} \tag{60}$$

$$\big(k(1-u) + G\big)(1+\mu) = (1-u)Q + b \tag{61}$$

$$(1+\mu) = E[(1-u)Q + b] \tag{62}$$

$$w = \nu\left[A - \frac{Tu}{(1-u)} + \frac{\xi\mu + \theta}{(1+\mu)}Q + \frac{\mu+\theta}{(1+\mu)(1-u)}b\right] \tag{63}$$

$$\begin{aligned} 0 &= \big[\rho + s - g - \frac{F}{N}\big]b + \mu\big[(A-w)(1-u) + \xi Q(1-u)\big] \\ &\quad + \Big[\big(k(1-u) + G\big)\frac{F}{N} + Tu\Big] \end{aligned} \tag{64}$$

$$0 = \big[\rho + s - g - \frac{F}{N} - \xi\mu\big]Q - (1+\mu)\big[A - w\big] \tag{65}$$

$$\frac{F}{N} = \beta\frac{(A-w)(1-u) + \xi(1-u)Q + b}{k(1-u) + G} \tag{66}$$

$$g = E(1-\beta)\Big[(A-w)(1-u) + \xi(1-u)Q + b\Big] \tag{67}$$

Before determining the bubbleless BGP values, let us highlight the relationship between the growth rate and the separation rate. From equations (60) and (61), we deduce that:

$$k(1-u)+G = \frac{1}{E} \tag{68}$$

Hiring cost G is positively correlated with time devoted to training u. Trained workers become more costly to recruit.

According to equations (57), (66) and (68) the separation rate value is written as:

$$s = E\beta\big[(A-w)(1-u)+\xi(1-u)Q\big] \tag{69}$$

From equation (67) and (69), we deduce that:

$$s = \frac{\beta}{1-\beta}g \tag{70}$$

When $b=0$, the bubbleless BGP values are derived by following these steps: Using (57), we extract $(A-w)$ from equation (65)

$$A-\tilde{w} = \big[\rho-\tilde{g}-\xi\mu\big]\frac{\tilde{Q}}{1+\mu} \tag{71}$$

One Substitute this equation into (67) to deduce that:

$$\tilde{g} = E(1-\beta)\big[(\rho-\tilde{g}-\xi\mu)\frac{1}{(1+\mu)}+\xi\big]\tilde{Q}(1-\tilde{u}) \tag{72}$$

Combining this last equation with (62), we derive the growth rate value:

$$\tilde{g} = \frac{1-\beta}{2-\beta}(\rho+\xi) \tag{73}$$

Once the value of $\tilde{g}$ is obtained, the other values follow naturally. Indeed, according to (70) and (73), the separation rate value is:

$$\tilde{s} = \frac{\beta}{2-\beta}(\rho+\xi) \tag{74}$$

$\tilde{r}$ is obtained from (55); equation (56) gives us $\tilde{T}$; $\tilde{u}$ is derived from (54); (62) provides us $\tilde{Q}$; the solution for $\tilde{G}$ is extracted from (68); $\tilde{\theta}$ is determined by (60); combining equations (57)-(59) we obtain $\tilde{N}$; (58) solves $\tilde{U}$; $\tilde{F}$ is derived from (57); and finally $\tilde{w}$ is obtained from (63).

A.4 Bubbly BGP solutions

Let us start with the determination of bubble value. From the credit constraint we know that:

$$\begin{aligned} [k^*(1-u^*)+G^*]F^*+T^*N^*u_t^* &= \left[A-w^*\right]N^*(1-u^*) \\ &\quad +\xi N^*(1-u^*)Q^*+N^*b^* \end{aligned} \tag{75}$$

Using this last equation with (57), then equation (64) becomes:

$$0 = \left[1+\rho-g^*\right]b^*+(1+\mu)\left[(A-w)(1-u)+\xi Q(1-u)\right] \tag{76}$$

We extract $(A - w)$ from equation (65)

$$A - w^* = \left[\rho - g^* - \xi\mu\right]\frac{Q^*}{1+\mu} \tag{77}$$

and substitute it into (76) to deduce that:

$$b^* = \frac{(1+\mu)[g^* - (\rho+\xi)]}{E(1-\xi)} \tag{78}$$

Involving the last equation, (67) is rewritten as:

$$g^* = E(1-\beta)\left[(A - w^*)(1-u^*) + \xi(1-u^*)Q^* + \frac{(1+\mu)[g^* - (\rho+\xi)]}{E(1-\xi)}\right] \tag{79}$$

We associate equations (77) and (62) with (79), we get the growth rate value:

$$g^* = \frac{(1-\beta)(\xi+\mu)}{(1-\beta)(\xi+\mu) - (1-\xi)}(\rho+\xi) \tag{80}$$

Therefore

$$b^* = \frac{(1+\mu)(\rho+\xi)}{E(1-\xi)}\left[\frac{(1-\beta)(\xi+\mu)}{(1-\beta)(\xi+\mu) - (1-\xi)} - 1\right] \tag{81}$$

The determination of the remaining bubbly BGP values is similar to that of the bubbleless BGP.