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A continuous approach to intergenerational preferences: sustainable preferences

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Abstract In the present paper we reflect and focus on the ideas of utility streams and welfare criterions from a point of view of social choice and sustainability, thinking out on the suitability of the corresponding mathematical definitions used in the literature in order to formalize these concepts. These terms were mathematically characterized on a discrete manner. However, a continuous approach is presented, such that it allows to apply some new techniques (related to mathematical and software tools) to the study of some social choice problems, in particular, for finding desired preferences satisfying some axioms such as Pareto, non-dictatorship or fairness on the set of continuous utility streams. As a consequence, we present and construct a fair sustainable preference.

Keywords Intergenerational preferences · sustainable development

1 Introduction and motivation

In several studies different variables x_i (such as availability of various resources, population, technological level, labour, capital,...) are considered in order to measure the welfare level of a generation and represent it just by a single number [1,2,5,7,12,17,18,19,20,21,23]. Each generation g is associated to its corresponding vector $(\alpha_1^g, ..., \alpha_n^g)$, where each coordinate α_i^g corresponds to

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the level of the variable x_i $(i = 1, ..., n)$ for the generation g. Then, a utility function u on $(\alpha_1^g, ..., \alpha_n^g)$ assigns a value α_g to each generation (we may assume that it is bounded by 0 and 1), such that it allows us to compare them.

Thus, if we associate each generation to a natural number, we may reduce our present study to the study of those utility streams $(\alpha_{g_k})_{k=1,2,...}$ on the space of infinitely bounded sequences l_∞. That is, we shall focus on the space of bounded utility streams, $\{(\alpha_{g_1}, \alpha_{g_2}, ..., \alpha_{g_k}, ...) : \alpha_{g_k} \in [0, 1], k \in \mathbb{N}\} \in l_\infty$, where each coordinate α_{g_k} represents the welfare level of the k-th generation.

One of the main problems in this field is the study of the 'best' utility stream, that is, the problem of choosing the best alternative from a pair of utility streams. For that purpose, the concept of *welfare criterion* is defined as an increasing function $W : l_\infty \to \mathbb{R}$, that must satisfy some desired axioms.

The classical approach to this problem carries some inconveniencies, some of them coming from the idea behind some definitions, others related to the mathematical overture.

The first drawback comes from the the notion of *generation*, which is not a mathematical concept at all. It refers to an arbitrary (an usually undefined) amount of years (10y, 15y, 20y,...), whose corresponding welfare level is represented then by a value in the corresponding vector's coordinate. Furthermore, even if the amount of years is fixed, the starting point of the counting may be different, so that in one study the first generation may start in 2000, for example, whereas in a second one it may start in 2004.

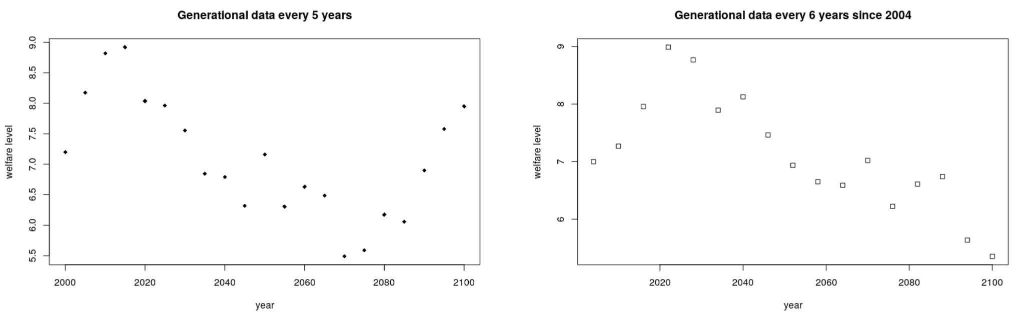

Fig. 1 Two possible data for different generations. In the first one the study focus on the time interval $[2000, 2100]$, with welfare data every 5 years. The second one goes from 2004 to 2100 and the welfare indices are represented every 6 years.

Therefore, the term is not the same in one study or in the other, and that may imply some inconveniences when dealing with the corresponding vectors (for example, if we want to compare or aggregate the information). Furthermore, given a utility stream such that there is a generation under the required minimum of welfare (usually defined by the satisfaction of basic needs [5]) but such that the welfare levels of generations before and after are well valued, we may wonder how this levels may change in the time and, in particular, how long the level was under the minimum. These questions appeal the study of that 'welfare path' that goes from one generation to another, so that even if we are working on a vector or set of points (generations), it may be useful to consider the curve or path that better approaches those points.

This is the first goal of the present paper, to improve the approach from a discrete scenario (sequence of generations) to a continuous one (time). For that, we understand and redefine the concept of utility stream through a continuous function from time to welfare levels.

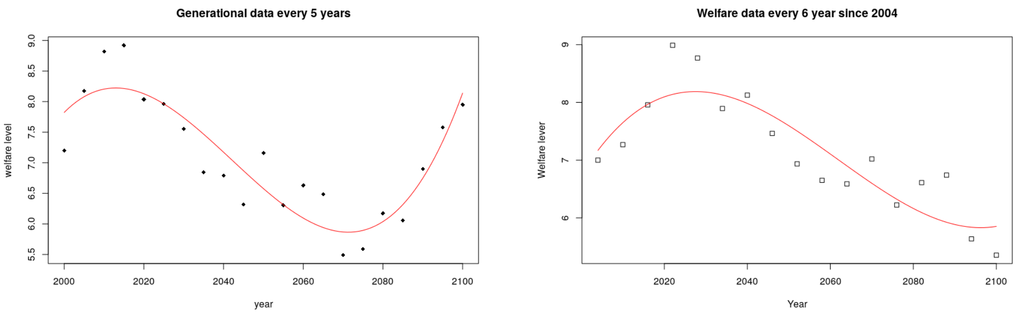

Fig. 2 A polynomical curve approximating the discrete data from the studies.

There is another important reflection that conditions deeply our study: the end of the Earth. In literature the study of intergenerational preferences has been always presented on the field of infinite vectors, that is, under the hypothesis of an infinite number of generations, always giving humanity (and in particular, planet Earth) a future. However, although the end must be far from now, it is well-known that the sun shall die in 5000 millions years, and the Earth (or what is left of her) will do it with it. It may seem a very long period of time (and in fact, it is, compared with human history, since prehistory it has passed a millionth of 5000 millions years), but it is bounded in any case, so it is much more far from the idea of eternity.

Furthermore, in literature, utility streams depend on several variables related to resources, population, pollution,... which are intrinsically related to the planet Earth. Hence, they are senseless without the existence of the planet. Thus, the existence and definition of these utility streams are conditioned to the existence of the Earth, which will disappear in 5000 millions years. Besides, it could seem unreal and extremely utopian to work with welfare preferences under the hypothesis that humanity will move from planet. In fact, even under this assumption, the only thing we could do for that success is just to guarantee a good welfare level in the final previous terrestrial phase.

This reflection may seem a little bit philosophical, however, it is important to deliberate about that, since the mathematical concepts and tools change from the bounded context to the unbounded one.

In addition to the reasons exposed in the introduction of the present paper, the big number of generations (in the classical term) that could hold in the life of the Earth also motivates us to represent the data through a function instead of by means of a huge set of ordered values. In the present paper we will deal with continuous functions $f\colon [0,1] \to [0,1]$, where the time interval $[0,1]$ may be interpreted as the period of time from present to the death of the sun (proportionally).

Thus, we propose to represent the data in a continuous way through functions from an age interval to welfare indices. Thus, instead of having vectors whose coordinates may fail to represent the same data, now we have functions on the same domain. Furthermore, several variables x_i (such as availability of various resources –oil, gas, water, oxygen,...–, population, pollution,...) that are considered in order to measure the welfare level of a generation may be continuous too. Then, we are able to aggregate the continuous data of the models, as it is shown in Figure 1, calculating the curve (in the model of the figure a polynomical curve of degree 3) that better approximates the mean of the predicted data by two models (notice that it is not possible to directly calculate the mean of the data, since they do not correspond to the same year, however we are able to make the predictions -every year- of two data in order to calculate then the mean).

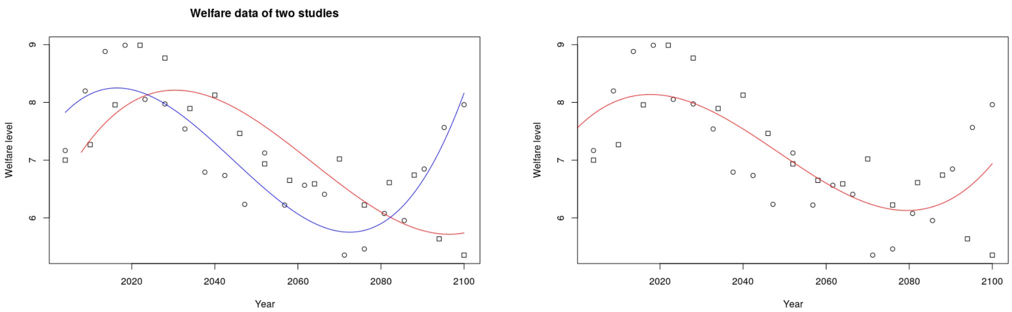

Fig. 3 Representation of both data together and the aggregation of both.

Even if (in practice) the data may be achieved in a discrete manner from time to time, these functions may be defined through programmes that construct the 'closest' function (under some reasonable criteria) to a set of points, that is, we interpolate data with the sole purpose of improving the study. Hence, even if the notion of generation differs, we would be able to traslate the information in a continuous manner, comparing and working with functions instead of infinite vectors. On the other hand, without loss of generality, we may reduce the domain of those functions to $I = [0,1]$, so that we work on the set of functions (we shall call them *continuous utility streams*) $\alpha\colon [0,1] \to [0,1]$, where $\alpha(t)$ represents the welfare level at the moment t (again, we may assume that the welfare level is represented with values between 0 and 1 or in any other desired real interval $[a,b]$).

Since the welfare level of a society rarely changes drastically, we will assume that these functions are continuous in all the interval. It can be generalized to functions which are continuous in all the interval except in a finite amount of points, in which an extreme event –war, natural disaster, rescue-plan, nuclear or chemical attack,...– happens that drastically changes the welfare level, but even in that case continuity assumption could be supported yet. Thus, and in order to simplify the main ideas, we shall work assuming the continuity of those functions. We shall denote the space of continuous real functions from $[0,1]$ to $[0,1]$ by $C([0,1])$.

Notice too that, when dealing with continuous functions $f\colon [0,1] \to [0,1]$ (i.e., integrable functions), we are able to measure the *total amount of welfare of f on a period* $P = [a,b]$ (we shall denote it by $TW(f)$) as well as the *average welfare of f on* $P = [a,b]$ (denoted by m_f) as follows:

$$TW(f) = \int_a^b f(t)dt; \qquad\qquad m_f = \frac{1}{b-a}\int_a^b f(t)dt.$$

Some other mathematical tools shall be useful too, such as deviation, measure or derivatives.

The second goal of the present work is to find a welfare criterion which satisfies some desirable axioms, for example, Pareto axiom (which is implicit in the definition of a welfare criterion), the axiom of non-dictatorship or some others related to the idea of *equity* [1,2,5,7,12,17,19,20,21,23]. The axioms satisfied by the welfare criterion defined in the present work are those that define the preference known as *sustainable preference* in [5]. Furthermore, we propose a mathematical definition in order to characterize the idea of *fairness*.

For this second study, it is necessary to redefine those concepts that where defined on the space of vectors or utility streams in l_∞. Concepts such as Pareto, non-dictatorship, Anonymity, distributivity fairness, the topology of the space, sensitivity to the present or future etc. have been studied in depth in the literature but on the discrete set l_∞ [1,2,5,7,19,20,23]. On the other hand, new concepts may arise that could be equivalent to the classical ones, since we may do the translation for the mathematical definition of the axiom on l_∞ or directly from the philosophical or essential idea that motivated the mathematical definition before.

2 Preliminaries

First, we introduce some basic definitions which are well-known and common in the literature of the field.

Definition 1 A *binary relation* $\mathcal{R}$ from A to X is a subset of the Cartesian product $A \times X$. In particular, in the case that $A = X$, the binary relation $\mathcal{R}$ is said to be defined on X, and it is a subset of the Cartesian product $X \times X$. Given two elements $a \in A$ and $x \in X$, we will use notation $a \mathcal{R} x$ to express that the pair (x, y) belongs to $\mathcal{R}$. Associated to a binary relation $\mathcal{R}$ from A to X, its *negation* is the binary relation $\mathcal{R}^c$ from A to X given by $(a, x) \in \mathcal{R}^c \iff (a, x) \notin \mathcal{R}$ for every $a \in A$ and $x \in X$.

Given two binary relations $\mathcal{R}$ and $\mathcal{R}'$ on a set X, it is said that $\mathcal{R}'$ *extends* or *refines* $\mathcal{R}$ if $x \mathcal{R} y$ implies $x \mathcal{R}' y$, that is, if $\mathcal{R}$ is contained in $\mathcal{R}'$.

The *transitive closure* of a binary relation $\mathcal{R}$ on a set X is the transitive relation $\mathcal{R}^+$ on set X such that $\mathcal{R}^+$ contains $\mathcal{R}$ and $\mathcal{R}^+$ is minimal.

The *transitive reduction* of a binary relation $\mathcal{R}$ on a set X is, in case it exists, the smallest relation having the transitive closure of $\mathcal{R}$ as its transitive closure.

Given a binary relation $\mathcal{R}$ on X, if two elements $x, y \in X$ cannot be compared, that is, $\neg(x \mathcal{R} y)$ as well as $\neg(y \mathcal{R} x)$, then it is denoted by $x \bowtie y$. We shall denote $x\mathcal{I}y$ whenever $x\mathcal{R}y$ as well as $y\mathcal{R}x$.

It is also very usual to denote $x \mathcal{R} y$ by $x \precsim y$. We shall use this notation for relations that appear at this work.

Definition 2 A *preorder* $\precsim$ on X is a binary relation on X which is reflexive and transitive. An antisymmetric preorder is said to be an *order*. A *total preorder* $\precsim$ on a set X is a preorder such that if $x, y \in X$ then $[x \precsim y] \vee [y \precsim x]$. A total order is also called a *linear order*, and a totally ordered set $(X, \precsim)$ is also said to be a *chain*. Usually, an order that fails to be total is also said to be a *partial order* and it is also denoted by $\preceq$. A subset Y of a partially preordered set $(X, \precsim)$ is said to be an *antichain* if $x \bowtie y$ for any $x, y \in Y$.

If $\precsim$ is a preorder on X, then the associated *asymmetric* relation or *strict preorder* is denoted by by $\prec$ and the associated *equivalence* relation by $\sim$ and these are defined, respectively, by $[x \prec y \iff (x \precsim y) \wedge \neg(y \precsim x)]$ and $[x \sim y \iff (x \precsim y) \wedge (y \precsim x)]$.

Definition 3 A total preorder $\precsim$ on X is called *representable* if there is a real-valued function $u\colon X \to \mathbb{R}$ that is order-preserving, so that, for every $x, y \in X$, it holds that $[x \precsim y \iff u(x) \leq u(y)]$. The map u is said to be a *utility function*.

In case of not necessarily total preorder, a real-valued function $u\colon X \to \mathbb{R}$ is said to be a *Richter-Peleg representation* if it satisfies that $[x \precsim y \Rightarrow u(x) \leq u(y)]$ (i.e. u is *isotonic*) as well as $[x \prec y \Rightarrow u(x) < u(y)]$. In case of a total preorder, this definition coincides with the previous one.

A preorder (not necessarily total) $\precsim$ on a set X is said to have a *multi-utility representation* (see [9]) if there exists a family $\mathcal{U}$ of isotonic real functions such that for any $x, y \in X$ the following holds true:

$$x \precsim y \Leftrightarrow \forall u \in \mathcal{U} \ u(x) \leq u(y).$$

Definition 4 The *space of bounded utility streams* is denoted and defined by $l_\infty = \{(\alpha_{g_1}, \alpha_{g_2}, ..., \alpha_{g_n}, ...) : \alpha_{g_n} \in [0,1], n \in \mathbb{N}\}$, where each coordinate α_{g_n} represents the welfare level of the n-th generation.

A *welfare criterion* is an increasing function $W\colon (l_\infty, \leq) \to (\mathbb{R}, \leq)$, that is, $\bar{x} \leq \bar{y}$ implies that $W(\bar{x}) \leq W(\bar{y})$, where the order $\leq$ on l_∞ is defined by

$$\bar{x} = (x_1, x_2, ..., x_n, ...) \leq (y_1, y_2, ..., y_n, ...) = \bar{y} \iff x_n \leq y_n, \text{ for any } n \in \mathbb{N}.$$

Hence, a welfare criterion defines a total preorder on the set of welfare streams, so that for any pair of utility streams $\alpha, \beta \in l_\infty$ it always holds that $W(\alpha) \leq W(\beta)$ or $W(\beta) \leq W(\alpha)$.

The following definitions are well known in the literature (see, e.g., [5]).

Definition 5 Let $\alpha = \{\alpha_g\}_{g=1,2,...} \in l_\infty$ be a stream and k an integer. The *k-th cutoff* of α is denoted by α^k and defined as follows:

$$\alpha^k = \{\sigma_g\}_{g=1,2,...}, \text{ where } \sigma_g = \alpha_g \text{ if } g \leq k, \text{ and } \sigma_g = 0 \text{ if } g > k.$$

Dually, the *k-th tail* of α is denoted by β_k and defined as follows:

$$\alpha_k = \{\sigma_g\}_{g=1,2,...}, \text{ where } \sigma_g = \alpha_g \text{ if } g > k, \text{ and } \sigma_g = 0 \text{ if } g \leq k.$$

Then, the pair (α^k, β_k) denotes the stream $\sigma = \alpha^k + \beta_k$ (here the sum is the usual vectors sums).

We include now the classical axioms related to dictatorship (see [5]).

Definition 6 Let W be a welfare criterion on l_∞. W is said to be a *dictatorship of the present* if for any two utility streams α and β, it holds that $W(\alpha) \leq W(\beta)$ if and only if there exists $N \in \mathbb{N}$ such that, if $k > N$, then $W(\alpha^k, \sigma_k) \leq W(\beta^k, \gamma_k)$, for all $\sigma, \gamma \in l_\infty$. This concept is naturally generalized to preferences on l_∞. We shall say that a welfare criterion satisfies *Axiom* 1 if there is no dictatorship of the present.

Dually, W is said to be a *dictatorship of the future* if for any two utility streams α and β, it holds that $W(\alpha) \leq W(\beta)$ if and only if there exists $N \in \mathbb{N}$ such that, if $k > N$, then $W(\sigma^k, \alpha_k) \leq W(\gamma^k, \beta_k)$, for all $\sigma, \gamma \in l_\infty$. This concept is naturally generalized to preferences on l_∞. We shall say that a welfare criterion satisfies *Axiom* 2 if there is no dictatorship of the future.

The goal of this paper is the study of sustainable preferences from a continuous point of view, characterizing the term by means of continuous utility streams from $[0,1]$ to $[0,1]$. The concept *sustainable preference* was defined as follows in [5].

Definition 7 A *sustainable preference* is a welfare criterion on l_∞ satisfying Pareto[1], Axioms 1 and 2.

[1] Chichilnisky uses the word the word *sensitive* (see [5]), which means that if a utility stream f is obtained from another g by increasing the welfare of some generations or periods

3 The continuous case: $C([0,1])$

In the present section we study a few concepts related to the field of welfare, as well as we propose a modification for them for the continuous case. By the way, we show some advantages is this new approach.

First, we reintroduce the definition of a welfare function, already explained in the introduction.

Definition 8 The *space of bounded and continuous streams* is the set of continuous functions from $[0,1]$ to $[0,1]$ and it is denoted by $C([0,1])$. In our context, a bounded and continuous stream $f\colon [0,1] \to [0,1]$ describes the welfare level of a society at an instant $t \in [0,1]$.

A *welfare function* is a function $W\colon C([0,1]) \to \mathbb{R}$ that assigns a real number $r \in \mathbb{R}$ to each bounded and continuous stream $\alpha \in C([0,1])$. We shall assume (without lost of generality) that welfare functions take values in $[0,1]$.

3.1 A little review on welfare criterions

Several welfare criterions have been defined in literature, some of them more graceful than others, at least from a mathematical point of view. In this section we collect some of them, as well as we modify them in order to adapt them to the continuous context.

The first concept we focus on is the so called *discounted sum of utilities*, which is defined as follows [5]:

Definition 9 A criterion $W\colon l_\infty \to \mathbb{R}$ is called *discounted sum of utilities* if it is of the form

$$W(\alpha) = \sum_{g=1}^{\infty} \lambda_g \alpha_g, \quad \forall \alpha \in l_\infty,$$

where $\lambda_g \geq 0$ and $\sum_{g=1}^{\infty} \lambda_g < \infty$.[2] The values λ_g are called by *discount factors*.

Remark 1 It is known (see [18]) that any discounted sum of utilities is a dictatorship of the present. Since the serie $\sum_{g=1}^{\infty} \lambda_g$ is convergent (that is, $\sum_{g=1}^{\infty} \lambda_g < \infty$) the sequence of weights $(\lambda_g)_{g\in\mathbb{N}}$ is decreasing and convergent to 0. Thus, the future is eventually weighted by 0 and therefore, this criterion is a dictatorship of the present.

We are able to generalize (and correct!) this idea to the continuous case as follows:

of time, then f is preferred to g, i.e. W must rank f strictly higher than g. This concept is also known as *Pareto*.

[2] Since the values α_g are bounded, the concept is well-defined.

Definition 10 A welfare function $W\colon C([0,1]) \to [0,1]$ is called *weighted integral of utilities* if it is of the form

$$W(\alpha) = \int_0^1 \lambda(t)\alpha(t)dt, \quad \forall \alpha \in C([0,1]),$$

where λ is an integrable and bounded function $\lambda\colon [0,1] \to \mathbb{R}$, that we shall call *discount function*.

Notice that the function λ defined for the weighing of the time may fail to be decreasing or convergent to 0. The concept of weighted integral of utility is well-defined for any integrable and bounded function $\lambda\colon [0,1] \to \mathbb{R}$. Hence, it does not necessarily imply dictatorship of the present either of the future.

Another criterion used in literature is the *Ramsey welfare criterion*. Again, as it holds with the discounted sum of utilities, this criterion is not well defined, since the series may diverge (see also [18]). We recover this criterion and adapt it to the continuous case, as well as we overcome that deficiency.

Here the classical definition:

Definition 11 *Ramsey's welfare criterion* ranks a utility stream $\alpha \in l_\infty$ above another β if α is closer to the constant function 1, that is, if

$$\sum_{g=0}^{\infty}(1-\alpha_g) \leq \sum_{g=0}^{\infty}(1-\beta_g).$$

And we modify that to the continuous case:

Definition 12 *Ramsey's welfare criterion* ranks a function $f \in C([0,1])$ above another g if f is closer to the constant function 1, that is, if

$$\int_0^1 (1-f(t))dt \leq \int_0^1 (1-g(t))dt.$$

This definition is equivalent to the next condition, which uses the concept of *total welfare amount*.

Definition 13 Let f be a function in $C([0,1])$. We define the *total welfare amount* of f (and denote it by $TW(f)$) by

$$TW(f) = \int_0^1 f(t)dt.$$

Proposition 1 Ramsey's welfare criterion *ranks a function f above another g if the total amount of welfare of f is bigger. That is, if*

$$\int_0^1 f(t)dt \geq \int_0^1 g(t)dt.$$

Proof Since $\int_0^1 (1-f(t))dt = \int_0^1 1\cdot dt - \int_0^1 f(t)\cdot dt$, and this is equal to $1-\int_0^1 f(t)\cdot dt$, we conclude that

$$\int_0^1 (1-f(t))dt \leq \int_0^1 (1-g(t))dt \iff \int_0^1 f(t)dt \geq \int_0^1 g(t)dt.$$

Remark 2 Notice that now, the Ramsey's welfare criterion is well defined, that is, there is always a number that corresponds to each continuous utility stream. That was not at all the case of the classical one, since the serie of the corresponding definition may fail to converge and, then, the value assigned to the utility stream would be ∞.

Now we focus on Rawlsian rule. This concept is well defined. Here we include the original definition as well as the corresponding adaptation to the continuous case.

Definition 14 *Rawlsian welfare criterion* ranks a utility stream $\alpha \in l_\infty$ above β if and only if $\inf\{\alpha_g\}_{g=1,2,...} > \inf\{\beta_g\}_{g=1,2,...}$.

For the continuous case, we say that *Rawlsian welfare criterion* ranks a continuous utility stream $f \in C([0,1])$ above g if and only if $\inf f = \inf\{f(t)\colon t \in [0,1]\} > \inf g = \inf\{g(t)\colon t \in [0,1]\}$.

Although in the present paper we only consider continuous functions, in case of dealing with not necessarily continuous utility streams, the definition above may be improved as follows:

Definition 15 *Rawlsian welfare criterion* ranks a function $f\colon [0,1] \to [0,1]$ above g if and only if $\inf_\mu f = \inf\{k\colon \mu\{f(t) \geq k\} > 0\} > \inf_\mu g = \inf\{k\colon \mu\{g(t) \geq k\}$, where μ is a measure on $[0,1]$.

The criterion related to satisfaction of basic needs is defined as follows.

Definition 16 The *criterion of satisfaction of basic needs* ranks a utility stream $\alpha \in l_\infty$ above β if and only if $T(\alpha) < T(\beta)$, where $T(\alpha)$ is defined by $T(\alpha) = \min\{t\colon \alpha_g \geq B, \forall g \geq t\}$ (here, B is the welfare level corresponding to the satisfaction of basic needs).

Remark 3 The criterion before seems coherent under the assumption that after achieving a satisfactory welfare level (that is, bigger than B) then this will not decrease below B again. Otherwise, given the following utility streams (with $B = 0.3$), $\alpha = (0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.4, 0.3, 0.2, 0.1, 0, 0.1, ...)$ (so, α is periodic) and $\beta = (0.1,\ 0.15, 0.2, 0.25, 0.3, 0.35,\ 0.4, 0.45, ..., 0.95, 1, 1, 1, ...)$, this criterion would rank α above β, which does not seem the best alternative.

Now, we modify this criterion for the continuous case and, by the way, we try to correct the drawback shown in remark before, so that now, if the (amount of) time without reach this basic level B in α is greater than in β, then this last one is preferred to α.

Definition 17 The *criterion of satisfaction of basic needs* ranks a continuous utility stream $\alpha \in C([0,1])$ above β if and only if $T(\alpha) < T(\beta)$, where $T(\alpha)$ is defined by $T(\alpha) = \mu\{t : \alpha(t) < B\}$.

Another criterion we will focus on is the so called *overtaking criterion*. It tries to rank a utility stream α over another β if α eventually leads to a permanently higher level of aggregate utility than does β (see [5]).

Definition 18 The *overtaking criterion* ranks a utility stream $\alpha \in l_\infty$ above β if and only if there exists $N \in \mathbb{N}$ such that for any $M > N$ it holds that $\sum_{g=1}^{M} \alpha_g \geq \sum_{g=1}^{M} \beta_g$.

Remark 4 The criterion before does not achieve its goal of ranking a utility stream α over another β if α eventually leads to a permanently higher level. To see that, we include the following case.

Consider the utility streams α and β such that $\alpha = (1, 0.5, 0.4, 0.3, 0.2, 0.1, 0, 0, 0, ...)$ and $\beta = (1, 1/2, 1/4, 1/8, ..., 1/2^n, ...)$. Then, taking $N = 6$, it holds that $2.5 = \sum_{g=1}^{M} \alpha_g > 2 \geq \sum_{g=1}^{M} \beta_g$ for any $M > N$, thus, according to that criterion, α is preferred above β. However, we may say that β eventually leads to a permanently higher level of aggregate utility than does α. Hence, our conclusion is that this criterion does not catch its objective. This handicap is corrected by the *long run average criterion*.

Now, we modify this criterion for the continuous case, but the deficiency explained in the remark before remains in for the continuous case too.

Definition 19 The *overtaking criterion* ranks a continuous utility stream $f \in C([0,1])$ above g if and only if there exists $T \in (0,1)$ such that for any $k > T$ it holds that $\int_1^k f(t)dt \geq \int_1^k g(t)dt$.

Finally, we recover the *long run average criterion*.

Definition 20 The *long run average criterion* ranks a utility stream $\alpha \in l_\infty$ above β if and only if there exist $N, K > 0$ such that $\frac{1}{T} \sum_{M}^{T+M} \alpha_g \geq \frac{1}{T} \sum_{M}^{M+T} \beta_g$, for any $T > N$, $M > k$.

In the continuous manner it would be as follows:

Definition 21 The *long run average criterion* ranks a continuous utility stream $f \in C([0,1])$ above g if and only if there exist $N, K \in (0,1)$ such that $\frac{1}{T} \int_a^{T+a} f(t) \geq \frac{1}{T} \int_a^{a+T} g(t)dt$, for any $T > N$, $a > k$ (with $a + T < 1$).

Remark 5 It is straightforward to see that in both definitions related to the long run average criterion, the fraction $\frac{1}{T}$ may be suppressed.

3.2 Axioms for sustainability in $C([0,1])$

Finally, after this little review on some welfare criterion, we focus on the axioms we will use for the continuous case. In the following lines we define them for the continuous case.

Definition 22 Let $\alpha\colon [0,1] \to [0,1]$ be a continuous stream and $k \in [0,1]$. The *k-th cutoff* of α is denoted by α^k and defined as follows:

$$\alpha^k(t) = \alpha(t), \text{ if } t \leq k, \text{ and } \alpha^k(t) = 0 \text{ if } t > k.$$

Dually, the *k-th tail* of α is denoted by β_k and defined as follows:

$$\beta_k = \alpha(t), \text{ if } t > k, \text{ and } \beta_k(t) = 0 \text{ if } t \leq k.$$

Then, the pair (α^k, β_k) denotes the stream $\sigma = \alpha^k + \beta_k$ (here the sum is the usual sum between functions).

Definition 23 Let W be a welfare function on $C([0,1])$. W is said to be a *dictatorship of the present* if for any two continuous utility streams α and β in $C([0,1])$, it holds that $W(\alpha) \leq W(\beta)$ if and only if there exists $T \in (0,1)$ such that, if $k > T$, then $W(\alpha^k, \sigma_k) \leq W(\beta^k, \gamma_k)$, for all $\sigma, \gamma \in C([0,1])$. This concept is naturally generalized to preferences on $C([0,1])$, i.e., a preference $\precsim$ on $C([0,1])$ is said to be a *dictatorship of the present* if for any two continuous utility streams α and β in $C([0,1])$, it holds that $\alpha \precsim \beta$ if and only if there exists $T \in (0,1)$ such that, if $k > T$, then $(\alpha^k, \sigma_k) \precsim (\beta^k, \gamma_k)$, for all $\sigma, \gamma \in C([0,1])$. We shall say that a welfare criterion (or a preference) satisfies *Axiom* 1 if there is no dictatorship of the present.

Dually, W is said to be a *dictatorship of the future* if for any two continuous utility streams α and β, it holds that $W(\alpha) \leq W(\beta)$ if and only if there exists $T \in (0,1)$ such that, if $k > T$, then $W(\sigma^k, \alpha_k) \leq W(\gamma^k, \beta_k)$, for all $\sigma, \gamma \in C([0,1])$. Again, we say that a preference $\precsim$ on $C([0,1])$ is said to be a *dictatorship of the future* if for any two continuous utility streams α and β in $C([0,1])$, it holds that $\alpha \precsim \beta$ if and only if there exists $T \in (0,1)$ such that, if $k > T$, then $(\sigma^k, \alpha_k) \precsim (\gamma^k, \beta_k)$, for all $\sigma, \gamma \in C([0,1])$. We shall say that a welfare criterion (or a preference) satisfies *Axiom* 2 if there is no dictatorship of the future.

Remark 6 Dictatorship condition may be rewritten by means of the strict relation $<$ (or $\prec$, for the case of a preference relation). In fact, W is a *dictatorship of the present* if for any two continuous utility streams α and β in $C([0,1])$, it holds that $W(\alpha) < W(\beta)$ if and only if there exists $T \in (0,1)$ such that, if $k > T$, then $W(\alpha^k, \sigma_k) < W(\beta^k, \gamma_k)$, for all $\sigma, \gamma \in C([0,1])$. Dually for a preference $\precsim$ on $C([0,1])$.

A welfare criterion W on l_∞ must be increasing or, in other words, it must satisfy Pareto's axiom. When dealing with continuous information instead of discrete, i.e. with continuous functions instead of vectors, the Pareto axiom could be directly defined by

$$f(x) \leq g(x),\ \forall x \in [0,1] \implies f \precsim g. \qquad (1)$$

However, we may improve the concept adapting it to the continuous case instead of directly translating the definition from the discrete case to the continuous one. Hence, using a measure μ on $I = [0,1]$ ($\mu([0,1]) = 1$), we can generalize it to the set of (not necessarily continuous) functions as follows:

Definition 24 Let f and g be two functions of X. We say that *Pareto* is satisfied if

$$\left\{ \begin{array}{l} \mu\{x \in I \colon g(x) < f(x)\} = 0, \\ \mu\{x \in I \colon f(x) \leq g(x)\} = 1, \end{array} \right\} \implies f \precsim g.$$

That is, $\mu\{x \in I \colon f(x) \leq g(x)\} = 1 \implies f \precsim g$.
Therefore,

$$\left\{ \begin{array}{l} \mu\{x \in I \colon f(x) \leq g(x)\} = 1, \\ \mu\{x \in I \colon f(x) < g(x)\} > 0. \end{array} \right\} \implies f \prec g.$$

We shall say that a welfare function satisfies Axiom 0 if it satisfies Pareto condition.

If the functions are supposed to be continuous, then the Pareto condition may be simplified to (1).

Proposition 2 *Let f and g be two continuous functions on X. Then, Pareto condition is satisfied if*

$$f(x) \leq g(x),\ \forall x \in [0,1] \implies f \precsim g.$$

Finally, we also define a concept which indicates the minimum welfare value.

Definition 25 The *minimum welfare value* of a (not necessarily continuous) utility stream f in $[0,1]$) is defined by

$$\min(f) = \inf\{k \colon \mu([f < k]) \neq 0\},$$

where $[f < k] = \{x \in I \colon f(x) < k\}$.

Remark 7 Under the hypothesis of the continuity of the function on $[0,1]$, then it is known (by Weierstrass Theorem) that there exists an absolute minimum.

The goal of this paper is the study of sustainable preferences from a continuous point of view, characterazing the term by means of continuous utility streams from $[0,1]$ to $[0,1]$.

Definition 26 A *sustainable preference* on $C([0,1])$ is a preorder satisfying Pareto and Axioms 1 and 2.

4 Fairness and Construction of a fair sustainable preference

In the present section we shall construct a sustainable preference on the space $C([0,1])$, that is, a preference satisfying Pareto condition and non-dictatorship. We will see that this is not enough in order to guarantee the idea of *equity* or *fairness* [1,2,19,23]. Hence, we shall define a fairness axiom and then, contruct a sustainable preference that also satisfies this new condition.

For this purpose, first we define the following relation.

Definition 27 The *Pareto preorder* $\precsim_p$ on $C([0,1])$ is defined as follows

$$f \precsim_p g \iff \mu\{t \in [0,1] \colon f(t) \leq g(t)\} = 1.$$

Remark 8 (1) Under the assumption of continuity, the Pareto preorder is characterized as follows:

$$f \precsim_p g \iff f(t) \leq g(t), \quad \text{for any } t \in [0,1].$$

(2) Notice that $f \precsim_p g$ implies $min(f) \leq min(g)$. Furthermore, since the utility streams are supposed to be continuous (i.e., $f, g \in C([0,1])$), $\precsim_p$ is antisymmetric. Hence, $f \sim_p g$ implies $f = g$, and $\precsim_p$ is actually a partial order.

Proposition 3 *The Pareto preorder $\precsim_p$ on $C([0,1])$ satisfies Axiom 0 as well as Axiom 1 and Axiom 2.*

Proof Axiom 0 is trivially satisfied by Pareto preorder. On the other hand, given $f, g \in C([0,1])$ such that $f \prec_p g$ and any $T \in (0,1)$, it is straighforward to see (see Remark 6) that the relation $(\alpha^k, \sigma_k) \prec_p (\beta^k, \gamma_k)$ (for all $\sigma, \gamma \in C([0,1])$, $k > T$) fails to be true. Thus, Axiom 1 is satisfied. We reason analogously for Axiom 2.

Since Pareto preorder satisfies those desired Axioms 0, 1 and 2, it may be thought that this relation is adequate and enough in order to correctly choose the best alternative on the set of continuous streams. However, since it is a partial and not total relation, it fails to compare some streams, even in cases in which the selection of the best alternative is trivial under the assumption of *fairness*, such as it is shown in the following example:

Example 1 Given the functions $f(x) = \sin(x) + 1$ and $g(x) = 1$ on the interval $[0, 4\pi]$, it is trivial that $f \bowtie_p g$. However, we may think that the stream g is preferred to f, since it is *fairer* (intergenerationally), whatever it means. To see that, notice that $TW(f) = TW(g) = 1$, but in f the welfare level of some generations are higher at the expense of others.

Due to the example before, we mathematically formalize the idea of *fairness*.

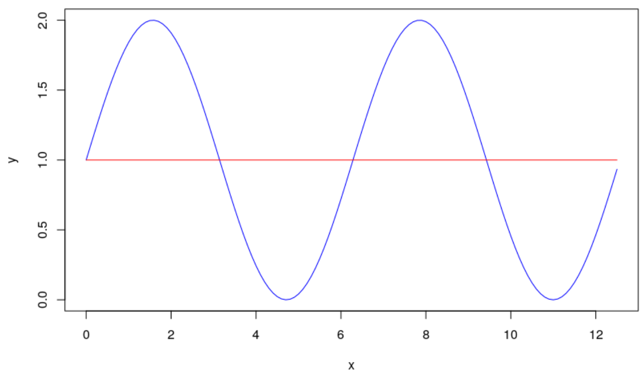

Fig. 4 Two continuous functions, $f(x) = \sin(x) + 1$ in blue and the constant function $g(x) = 1$ in red, on the interval $[0, 4\pi]$.

Definition 28 A relation $\precsim$ on $C([0,1])$ is said to fatisfy the *fairness* axiom if

$$\{f \precsim g \quad \text{and} \quad var(f) < var(g)\} \implies f \prec_p g.$$

We shall rename the fairness axiom by *Axiom 3*.

Remark 9 (1) Here, the deviation of the stream measures the inequality between generations with respect to the mean. Hence, under the assumption of fairness, we accept a higher level of inequality just when the welfare level of each generation is also higher, i.e., when the alternative chosen is preferred with respect to Pareto relation, as it is shown in Example 2.
(2) From an applied point of view, the corresponding deviation may be calculated directly from the collected data or from the continuous function that approach all those points. We define the deviation of a continuous function f on $[a, b]$ by

$$var(f) = \frac{1}{b-a}\int_a^b (f(x) - m_f)^2 dx,$$

where m_f is the corresponding mean of f, that is, $m_f = \frac{1}{b-a}\displaystyle\int_a^b f(x)dx$.

Example 2 Given the functions $f(x) = \sin(x) + 3$ and $g(x) = 1.5$ on the interval $[0, 4\pi]$, it is trivial that $var(g) = 0 < var(f)$. However, the welfare level described by f is always greater than the level represented by g, that is, $g \prec_p f$. Thus, we conclude that the stream f is preferred to g, and this election satisfies the aforementioned fairness axiom.

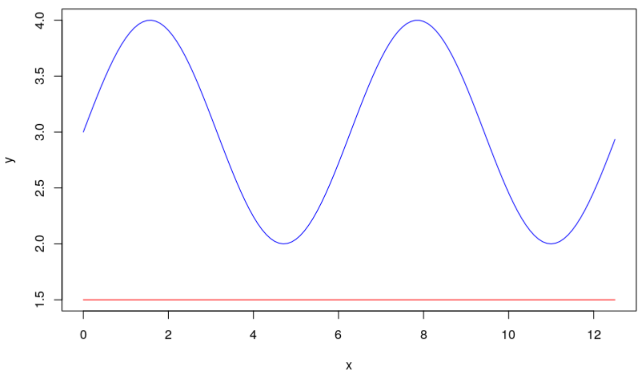

Fig. 5 Two continuous functions, $f(x) = \sin(x) + 3$ in blue and the constant function $g(x) = 1'5$ in red, on the interval $[0, 4\pi]$.

Fairness condition is trivially satisfied by Pareto relation, however, this relation gives no election in some incomparable (with respect to Pareto relation) cases in which the best alternative is clear, such as is Example 2. Therefore, we conclude this study through a refinament of the Pareto relation.

Definition 29 The preorder $\precsim_v$ on $C([0,1])$ is defined as follows by means of the deviation, the minimum and the amount of welfare:

$$f \precsim_v g \iff \{var(g) \leq var(f),\, min(f) \leq min(g) \text{ and } \int_0^1 f(t)dt \leq \int_0^1 g(t)dt\}.$$

In other words, if we define a function v from the space of continuous utility streams $C([0,1])$ to $[0,1]^3$ by $v(f) = (1 - var(f), min(f), \int_0^1 f(t)dt)$, then it holds that $f \precsim_v g$ if and only if $v(f) \leq v(g)$.

Remark 10 It is possible to construct a multi-utility representation $\mathcal{U}$ of $\precsim_v$ just by means of three functions $\mathcal{U} = \{1 - var, \min, TW\}$, where var, min and TW are the deviation, the minimum and the total welfare of the corresponding stream (notice here that the value of the deviation of a stream $f \in C([0,1])$ is smaller than 1).

Proposition 4 *The preorder $\precsim_v$ on $C([0,1])$ satisfies Axiom 1, Axiom 2 and Axiom 3, but not Axiom 0.*

Proof Axiom 3 is trivially satisfied. Again, given $f, g \in C([0,1])$ such that $f \prec_p g$ and any $T \in (0,1)$, since the values of var, min and TW would change, it is straighforward to see (see Remark 6) that the relation $(\alpha^k, \sigma_k) \prec_p (\beta^k, \gamma_k)$

(for all $\sigma, \gamma \in C([0,1])$, $k > T$) fails to be true. Thus, Axiom 1 is satisfied. We reason analogously for Axiom 2.

Example 4 shows that $\precsim_v$ fails to satisfy Pareto.

Now, we construct a preorder on the space $C([0,1])$ satisfying the desired axioms of Pareto, non-dictatorship and fairness (that is, Axioms 0,1, 2 and 3).

Definition 30 The relation $\precsim_w$ on $C([0,1])$ is defined as the transitive closure of the union $\precsim_p \cup \precsim_v$, that is, $\precsim_w = (\precsim_p \cup \precsim_v)^+$.

Remark 11 According to Definition 30, given two continuous utility streams g and h, if there exists a family $(f_k)_{k=1}^n$ of continuous streams (for some $n \in \mathbb{N}$) such that $g \precsim_* f_1 \precsim_* \cdots \precsim_* f_n \precsim_* h$ (where $\precsim_*$ may be $\precsim_p$ and $\precsim_v$), then $g \precsim_w h$.

We may found cases in which $f \bowtie_v g$ as well as $f \bowtie_p g$, as it is shown in the next example.

Example 3 Given the functions $f(x) = \sin(x)+2$ and $g(x) = 1'5$ on the interval $[0, 4\pi]$, we may calculate and conclude that:

1. $\min(f) < \min(g)$
2. $TW(g) = \int_0^{4\pi} g(x)dx < \int_0^{4\pi} f(x)dx = TW(f)$
3. $var(g) = \frac{1}{4\pi}\int_0^{4\pi}(g(x) - m_g)^2 dx < \frac{1}{4\pi}\int_0^{4\pi}(f(x) - m_f)^2 dx = var(f)$, where m_g and m_f are the corresponding means of g and f, respectively.

Therefore, we deduce that $f \bowtie_v g$. Besides, it is obvious that $f \bowtie_p g$ so, we conclude that $f \bowtie_w g$.

There are also cases in which $f \bowtie_v g$ as well as $g \prec_p f$, as it is shown in the next example.

Example 4 Given the functions $f(x) = \sin(x)+3$ and $g(x) = 1'5$ on the interval $[0, 4\pi]$, we may calculate and conclude that:

1. $\min(g) < \min(f)$
2. $TW(g) = \int_0^{4\pi} g(x)dx < \int_0^{4\pi} f(x)dx = TW(f)$
3. $var(g) = \frac{1}{4\pi}\int_0^{4\pi}(g(x) - m_g)^2 dx < \frac{1}{4\pi}\int_0^{4\pi}(f(x) - m_f)^2 dx = var(f)$, where m_g and m_f are the corresponding means of g and f, respectively.

Therefore, we deduce that $g \bowtie_v f$. Besides, it is obvious that $g \prec_p f$ so, we conclude that $g \prec_w f$.

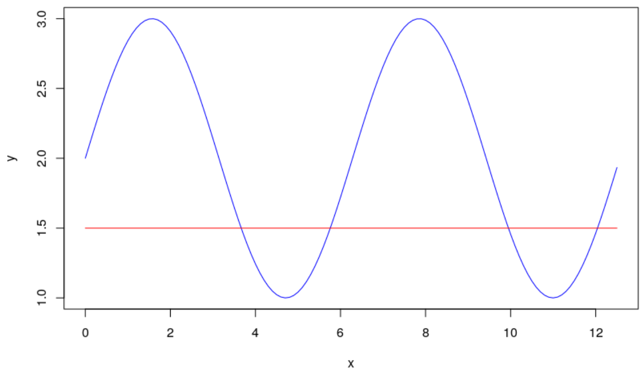

Fig. 6 Two continuous functions, $f(x) = \sin(x) + 2$ in blue and the constant function $g(x) = 1'5$ in red, on the interval $[0, 4\pi]$.

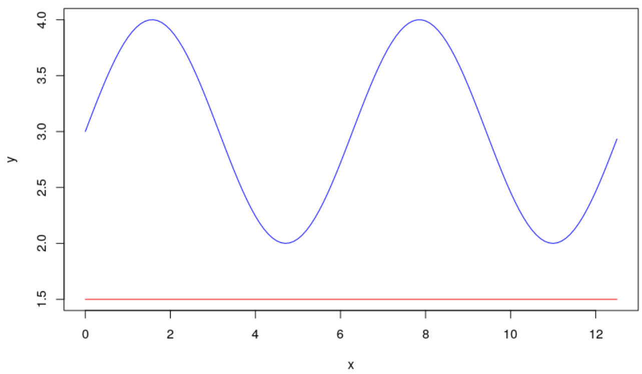

Fig. 7 Two continuous functions, $f(x) = \sin(x) + 3$ in blue and the constant function $g(x) = 1'5$ in red, on the interval $[0, 4\pi]$.

Now, in the following example, we show a case in which $f \prec_v g$ as well as $g \bowtie_p f$.

Example 5 Given the functions $f(x) = \sin(x + \pi/2) + 2$ and $g(x) = \frac{x}{2\pi} + 2$ on the interval $[0, 4\pi]$, we may calculate and conclude that:

1. $\min(f) < \min(g)$
2. $TW(f) = \int_0^{4\pi} f(x)dx < \int_0^{4\pi} g(x)dx = TW(g)$
3. $var(f) = \frac{1}{4\pi} \int_0^{4\pi} (f(x) - m_f)^2 dx > \frac{1}{4\pi} \int_0^{4\pi} (g(x) - m_g)^2 dx = var(g)$, where m_g and m_f are the corresponding means of g and f, respectively.

Therefore, we deduce that $f \prec_v g$. Besides, it is obvious that $g \bowtie_p f$ so, we conclude that $f \prec_w g$.

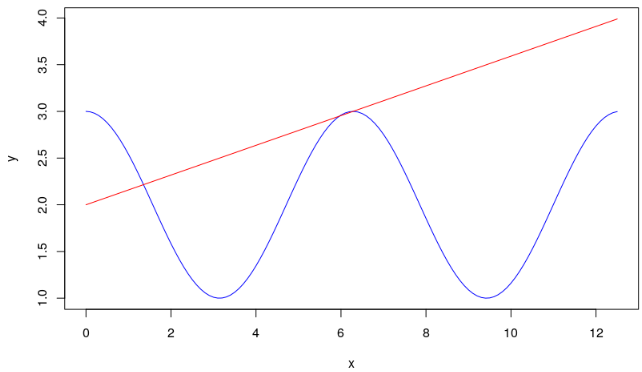

Fig. 8 Two continuous functions, $f(x) = \sin(x + \pi/2) + 2$ in blue and $g(x) = \frac{x}{2\pi} + 2$ in red, on the interval $[0, 4\pi]$.

Finally, we include an example in which $f \prec_v g$ as well as $f \prec_p g$.

Example 6 Given the functions $f(x) = \frac{1}{2}\sin(x) + 1$ and $g(x) = 2$ on the interval $[0, 4\pi]$, we may calculate and conclude that:

1. $\min(f) < \min(g)$
2. $TW(f) = \int_0^{4\pi} f(x)dx < \int_0^{4\pi} g(x)dx = TW(g)$
3. $var(g) = \frac{1}{4\pi}\int_0^{4\pi}(g(x) - m_g)^2 dx > \frac{1}{4\pi}\int_0^{4\pi}(f(x) - m_f)^2 dx = var(f)$, where m_g and m_f are the corresponding means of g and f, respectively.

Therefore, we deduce that $f \prec_v g$. Besides, it is obvious that $f \prec_p g$ so, we conclude that $f \prec_w g$.

Theorem 1 *The relation* $\precsim_w = (\precsim_p \cup \precsim_v)^+$ *is a partial preorder.*

Proof By definition, $\precsim_w$ is transitive. Let's see that there are no cycles, that is, that $f \prec_w g \prec_w f$ does not hold.

By contradiction, suppose that $f \prec_w g$ as well as $g \prec_w f$. By Definition 30, if it holds that $f \prec_w g$, then $min(f) \leq min(g)$ and $TW(f) \leq TW(g)$ are satisfied. Dually, if it holds $g \prec_w f$, then $min(g) \leq min(f)$ and $TW(g) \leq TW(f)$, hence $min(f) = min(g)$ and $TW(f) = TW(g)$.

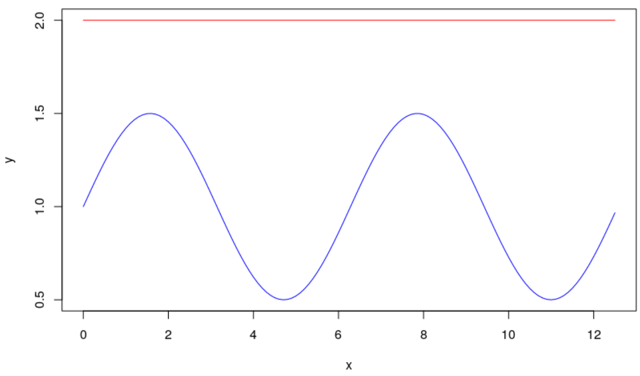

Fig. 9 Two continuous functions, $f(x) = \frac{1}{2}\sin(x) + 1$ in blue and $g(x) = 2$ in red, on the interval $[0, 4\pi]$.

Thus, since $\prec_p$ implies a difference in the total welfare, for any family $(f_i)_{i=1}^n$ of continuous utility streams such that $f \precsim_* f_1 \precsim_* \cdots \precsim_* f_n \precsim_* g$, there is no k such that $f \precsim_* f_1 \precsim_* \cdots \precsim_* f_{k-1} \prec_p f_k \precsim_* \cdots \precsim_* f_n \precsim_* g$. Hence, by Remark 8, we conclude that in the chain $f \precsim_* f_1 \precsim_* \cdots \precsim_* f_n \precsim_* g$ all the relations $\precsim_*$ are $\precsim_v$. Thus, we conclude that $f \prec_v g$. We argue dually with $g \prec_w f$ arriving to the desired contradiction: $f \prec_v g$ as well as $g \prec_v f$.

Theorem 2 *The relation $\precsim_w$ is a partial preorder on $C([0,1])$ satisfying Axioms 0, 1, 2 and 3. Hence, it is a fair sustainable preference.*

Proof By Theorem 1 it is clear that $\precsim_w$ is a partial preorder on $C([0,1])$. Since $\precsim_w = (\precsim_p \cup \precsim_v)^+$, it is clear too that Axiom 0 (related to Pareto condition) is also satisfied.

From Proposition 3 and Proposition 4 we deduce that Axioms 1 and 2 are satisfied. Fairness condition is trivially satisfied by Pareto relation, as well as by $\precsim_v$ by Proposition 4. Thus, $\precsim_w$ is a fair partial preorder so, we conclude that $\precsim_w$ is a fair sustainable preference.

5 Representability

In the previous section a sustainable and fair preference relation has been introduced in Definition 30. Since this preference is in fact a preorder on $C([0,1])$, it is well-known (see [9]) that it admits a multi-utility representation. It is also well-known that, by Szpilrajn extension theorem [22], any preorder can be extended to a linear order. However, it is not possible to guarantee the finiteness of the family of functions of a multi-utility, in particular when the preorder fails to be near-complete [9], and in the other hand, the linear extension is not always representable. Hence, the representability problem and the study of the possible extensions of the aforementioned sustainable and fair preference relation $\precsim_w$ is not trivial. In the present section we includes a few ideas related to this problem.

Proposition 5 *Let $\precsim_p$ be the Pareto preorder. Then, the following relation $\leq_{tw}$ defined by*

$$f \leq_{tw} g \iff v_3(f) \leq v_3(g), \quad f, g \in C([0,1]),$$

is a linear extension of $\precsim_p$, where $v_3(f) = TW(f)$, for any $f \in C([0,1])$.

Proof It is trivial that $\leq_{tw}$ is a total preorder. If $f \precsim_p g$ then $v_3(f) \leq v_3(g)$. Therefore, $f \precsim_p g$ implies $f \leq_{tw} g$ and, hence, $\leq_{tw}$ extends $\precsim_p$.

Proposition 6 *Let $\precsim_w$ be the sustainable preference defined on Definition 30. Let $p_2, p_3 \geq 0$ be two weights related to the minimum and the total welfare. Then, the following relation $\leq_{23}$ defined by*

$$f \leq_{23} g \iff \sum_{i=2}^{3} p_i \cdot v_i(f) \leq \sum_{i=2}^{3} p_i \cdot v_i(g), \quad f, g \in C([0,1]),$$

is a linear extension of $\precsim_w$, where $v_2(f) = \min(f)$ and $v_3(f) = TW(f)$, for any $f \in C([0,1])$.

Proof It is trivial that $\leq_{23}$ is a total preorder. If $f \precsim_w g$ then $v_3(f) \leq v_3(g)$ as well as $v_2(f) \leq v_2(g)$. Therefore, $f \precsim_w g$ implies $f \leq_{23} g$ and, hence, $\leq_{23}$ extends $\precsim_p$.

Remark 12 In the extension before it is not trivial to include a function related to the deviation (such as v_1) since it is possible to hold true $f \prec_w g$ as well as $var(f) << var(g)$ at the same time, for some $f, g \in C([0,1])$, as it was shown in Example 2.

It would be interesting to define a linear extension including the deviation. That is, we wonder if there exist $p_1, p_2, p_3 > 0$ (maybe under some suitable conditions) such that

$$f \leq_w g \implies \sum_{i=1}^{3} p_i \cdot v_i(f) \leq \sum_{i=1}^{3} p_i \cdot v_i(g), \quad f, g \in C([0,1]).$$

We leave this some other questions for future works, and we close this section with the following proposition.

Proposition 7 *Let $\precsim_w$ be the sustainable preference defined on Definition 30. Let ϕ be a function on $C([0,1]) \times C([0,1])$ such that $\phi(f,g) = 1$ if $f \prec_p g$ and 0 otherwise. Let $p_1, p_2, p_3 \geq 0$ be three weights related to the deviation, the* *minimum and the total welfare such that* $p_1 + p_2 + p_3 = 1$*. Then, the following relation* $\leq_w$ *defined by*

$$f \leq_w g \iff \frac{1}{3}\sum_{i=1}^{3} p_i \cdot v_i(f) \leq \frac{1}{3}\sum_{i=1}^{3} p_i \cdot v_i(g) + \phi(f,g), \quad f,g \in C([0,1]),$$

is a linear extension of $\precsim_w$*, where* $v_1(f) = 1 - var(f)$*,* $v_2(f) = \min(f)$ *and* $v_3(f) = TW(f)$*, for any* $f \in C([0,1])$*.*

Proof It is trivial that $\leq_w$ is a total preorder. If $f \precsim_v g$ then $v_i(f) \leq v_i(g)$ for any $i = 1,2,3$. On the other hand, if $f \precsim_p g$ then $v_3(f) \leq v_3(g)$ as well as $v_2(f) \leq v_2(g)$, furthermore, it also holds that $|v_1(f) - v_1(g)| \leq 1 = \phi(f,g)$. Thus, in both cases $\frac{1}{3}\sum_{i=1}^{3} p_i \cdot v_i(f) \leq \frac{1}{3}\sum_{i=1}^{3} p_i \cdot v_i(g) + \phi(f,g)$ holds true. Hence, $f \precsim_w g$ implies $f \leq_w g$.

6 Further comments

The study developed in this paper has been made under some criteria such as *total welfare*, *minimum* or *deviation* that actually define a total preorder on the set of functions $C([0,1])$. The Pareto criterion fails to be total but it also defines a transitive relation on the set.

However, throughout the paper the reader may think that some criterias may be improved. For instance, one may suggest that function f of Figure 6 should be preferred to function g, even when the minimun of f is smaller than that of g. Hence, it could seem admissible to define a small threshold such that some little bit worse levels of welfare for a few generations may be preferred at the expense of great improvements for others. Similarly, a small but bigger social inequality may be accepted if it returns in some important benefits.

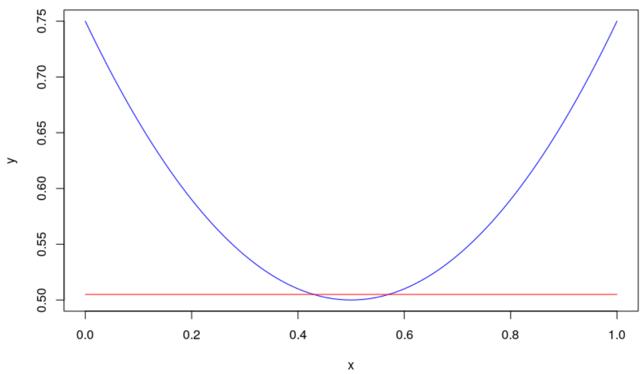

Fig. 10 Two continuous functions, $f(x) = (x - \frac{1}{2})^2 + 0'5$ in blue and $g(x) = 0'505$ in red, on the interval $[0,1]$.

These kind of dilemas may be discussed and modelized by means of thresholds or functions. Hence, instead of working just with transitive relations such as preorders (as it has been done throughout this paper), some other intransitive relations will appear when modeling these dilemas, for instance, interval orders and, in particular, semiorders in case of dealing with thresholds. [3,10, 11,16]

References

1. Banerjee K. and Mitra T. (2008) *On the continuity of ethical social welfare orders on infinite utility streams.* Social Choice and Welfare, **30**, 1-12.
2. Basu K., Mitra T. (2003) *Aggregating infinigte utility streams with intergenerational equity: the impossibility of being paretian.* Econometrica, **71** (5) , 1557-1563.
3. D.S. Bridges and G.B. Mehta (1995) *Representations of Preference Orderings*, Berlin-Heidelberg-New York: Springer-Verlag.
4. Candeal J.C., Chichilnisky G., Indurain E. (1997) *Topological aggregation of preferences:the case of a continuum of agents*, Social Choice and Welfare **14** 333-343.
5. Chichilnisky G. (1996) *An axiomatic approach to sustainable development*, Social Choice and Welfare **13** 231-257.
6. Hart K.P., Jun-iti Nagata, Vaughan J.E. (2003) *Encyclopedia of General Topology*, 1st Edition. ISBN: 9780444503558.
7. Diamond P. (1965) *The evaluation on infinite utility streams*, Econometrica **33** 170-177.
8. Estevan A., Valero O. *Intergenerational Preferences and Continuity: Reconciling Order and Topology.* Submitted to RACSAM.
9. Evren O., Ok E.A. (2011) *On the multi-utility representation of preference relations*, Journal of Mathematical Economics **47** 554-563.
10. Fishburn P.C. (1970) *Intransitive indifference with unequal indifference intervals*, Journal of Mathematical Psychology **7** 144-149.
11. Fishburn P.C. (1970) *Intransitive indifference in preference theory: a survey*, *Opererations Research* **18(2)** 207-228.
12. Hartwick J.M. (1977) *Intergenerational Equity and the Investing of Rents from Exhaustible Resources*, The American Economic Review, **67 (5)** 972-974.
13. Kunzi H.P.A. *Nonsymmetric distances and their associated topologies: About the origins of basic ideas in the area of asymmetric topology*, Handbook of the History of General Topology, Aull, C.E. and Lowen R. (Eds.), vol. 3, (Kluwer, Dordrecht, 2001), 853-968.
14. V. L. Levin, *Functionally closed preorders and strong stochastic dominance*, Soviet Math. Doklady **32** (1985), 22-26.
15. Levin V. L. (2001), *The Monge-Kantorovich problems and stochastic preference relation*, Advances in Mathematical Economics, **3** 97-124.
16. Luce R.D. (1956) Semiorders and a theory of utility discrimination, *Econometrica* **24** 178-191.
17. Pezzey J.C.V. (1997) *Sustainability Constrains versus 'Optimality' versus Intertemporal Concern, and Axioms versus Data*, Land Economics **73 (4)** 448-466.
18. Pezzey J. C. V., Toman M. A. (2002) *The Economics of sustainability: A Review of Journal Articles.* Discussion Papers from Resources For the Future. Ashgate Dartmouth.
19. Sakai T. (2003) *An axiomatic approach to intergenerational equity*, Social Choice Welfare, **20** 167-176.
20. Sakai T. (2003) *Intergenerational preferences and sensitivity to the present*, Economics Bulletin, **4**, (26) 1-05.
21. Solow R.M. (1974) *Intergenerational Equity and Exhaustible Resources*, The Review of Economic Studies Symposium, 29-45.
22. Szpilrajn E. (1930) Sur l'extension de l'ordre partiel, Fundamenta Mathematicae, **16** 386-389.
23. Svensson L.G. (1980) *Equity Among Generations*, Econometrica **48**, 1251-1256.