On rational choice from lists of sets

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Abstract

We analyze the *rationality* of a Decision Maker (DM) who chooses from *lists of sets of alternatives.* A new class of choice functions, representing DM's choice-behavior, and a novel rationality axiom are proposed and studied.

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1 Introduction

The present paper is devoted to study the *rationality* of a Decision Maker (DM) who chooses from *lists of sets* of alternatives, where for a list $\mathcal{A} = (A_1, \ldots, A_k)$ we mean a finite collection of *disjoint* sets whose order matters. We propose considering the DM's behavior as *rational* if she never selects previously disregarded alternatives from a given list. This quite obvious consideration is the key principle on which the new rationality axiom, that we propose and call *No-Regret* (**NR**), relies.

We show that **NR** encompasses some of the most prominent rationality notions of the classical choice model, which assumes the DM chooses from a set of *mutually exclusive* alternatives and, in particular, it generalizes a sort of *independence of irrelevant alternatives* property to the present more general setting.

We observe there are a variety of real life decisions requiring selections to be made from sets of alternatives presented according to some specific order. Some possible examples are:

1. A DM, that manages a venture capital fund, chooses some bonds, stocks, equities from the set A_t of all the securities available at each moment t, with $t = 1, \ldots, n$. Alternatively, t could denote different stock exchanges (for instance, Singapore, New York, London etc.) ordered in a list according to their opening time and from which the DM chooses different securities.

2. A DM chooses from a list of sets of wines collected according to their characteristics (white, red, dessert wine etc.)

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3. A DM wants to buy a watch and she chooses from an online list of sets of watches grouped by brand name (Rolex, Breguet, IWC, etc.).
4. In an exam, a student has to solve two problems from a list of sets of exercises divided by subject.
5. A customer has to fill a satisfaction survey selecting from a list of questions her level of satisfaction.
6. A scholar sequentially checks a list of scientific journals looking for new published papers on the topic she is studying.
7. Consider the problem of finding an equilibrium in a market with excess demand. A DM with endowment ω faces prices p, making a choice x_1 from the budget set $A_1 := \{x : (p^1, x) \leq (p^1, \omega)\}$ at the current prices p^1. The choice of the DM (and other agents), changes prices in order to diminish the excess demand. Afterwards, she makes a choice from $A_t = \{x : (p^t, x) \leq (p^t, x_{t-1})\}$, with $t = 2, \ldots k$, following a kind of *iterative procedure* in which the budget set at time t depends on the choice (and equilibrium prices) at $t-1$. The DM chooses $(x_1, \ldots, x_k)$ from the list $\mathcal{A} = (A_1, \ldots, A_k)$. We observe that a DM might manipulate her endowments in order to get a better equilibrium output at the end of the choice procedure, adopting, in this way, a non-rational behavior.
8. The coach of the national team has to select k players for the next World Cup from the 8 teams that play a knockout-tournament in the national competition. There are four quarterfinals, with the losers eliminated, followed with the semifinals, with the losers playing an extra match to decide the bronze medal. The winner of the final gets the gold medal, the loser the silver. He selects players from teams according to two criteria: players with fresher legs (hence, those whose team lost the match at the quarterfinals) and strongest players (presumably those who won the final).

These non-exhaustive list of examples of choice from lists of sets could be integrated by many other instances from commonly observed situations as e.g. the online shopping from different departments or the hospital's hiring process of doctors grouped by specialization. Nonetheless, our interest is not motivated by any of these specific examples, the main aim of the present work is in fact more in general the study of the *rationality* of a DM who has to solve her decision problem of choice from lists of sets of alternatives.

We consider the DM's behavior as *rational* if she does not choose earlier ignore alternatives that are worthless for her (**NR** axiom). This means that the not-chosen alternatives can be removed from the set from which the DM chooses without any impact on its value (*Outcast* property, **O**). We show that **NR** axiom encompasses both the Heritage axiom (**H**), (a fundamental rationality property, saying that if an alternative is chosen from a set it will be chosen from any subset of the latter containing it) and the Path-independent axiom (**PI**) (according to which the DM's choice does not depend on any particular order of the alternatives). We study the relationships between **NR** property and these three (**H**, **O**, **PI**) classical notions of rationality with the *aim* to test *by comparison* that the new **NR** property is a quite general rationality axiom.

We consider the choice from lists of sets either as (i) a subset of the union of all the sets of a list or

as (*ii*) a sub-list of the list from which the DM chooses. We show that the class of choice functions on lists of sets of alternatives (**CFL**) that satisfy the **NR** property is *stable under union*, but not stable under intersection (see Section 2 below).
Then, we proceed by studying four general mechanisms of choice from lists of sets. In the first three, a DM chooses from a list *facing one set of alternatives at a time*, so she credibly acts according to the classical choice model, namely her decision making behavior can be captured by a choice function selecting alternative(s) from a set. In the forth, the choice mechanism is induced by a binary relation over sets (*hyperrelation*). Thus
(1) We first analyze an *iterative search mechanism* (see among others Masatlioglu and Nakajima, (2013)) that typically arises whenever, for instance, a DM buys a book from an online book-shop. Next time she looks for a new book from the same web-store, she will see a set of books suggested by the algorithm that takes into account the previous purchase and that in addition suggests her to buy again the first book itself. We show (see Theorem 1) that a DM that, according to our example, does not buy books that she disregarded as boring and uninteresting is endowed with the *pseudorationality* (see Moulin, (1985)), i.e. her choice is not sensitive to any particular order of presentation of the different sets of alternatives in a list when her choice mechanism is of iterative-search type.
(2) Then, we consider the issue of solving the smaller parts of a complex choice problem *sequentially* in order to gain a better understanding of the problem: this mental technique has been applied in mathematics, logic, and in decision processes, since before Aristotle. In the cognitive sciences, Newell and Simon (1971) have shown that one mental strategy for solving a complicated problem is to analyse parts of it *sequentially* so as to minimize dependencies between parts and maximize the possibility of obtaining the best solution. In economics, people *sequentially* choose from a *list of sets* into which a set of alternatives has been divided. For instance, in many-to-many matching models (see e.g. Aygun and Sonmez (2012), Echenique and Oviedo (2006) and Roth and Sotomayor (1990)), colleges are faced with sets of candidates partitioned into groups according to their preferences for the colleges. Namely, candidates who have one college as first choice are grouped in the first set, others for whom that college is the second choice are in the second set of candidates and so on. Candidates thus partitioned form lists of sets of alternatives from which colleges are called upon to choose. A selection committee choosing *sequentially* from those sets decides to offer a place to some candidates, who therefore represent the choice from a list of sets of alternatives. In order to provide a rationale for this issue, we construct a **CFL** by implementing the classical choice function f that selects *sequentially* from the element-sets of a list. We show such a choice function on lists satisfies **NR** if and only if the choice function f inducing such **CFL** satisfies the Path-independent axiom (see Theorem 2 below).

We remark here the importance of such first two results that can be interpreted as new characterizations of the much studied class of path-independent choice functions. Therefore, **NR** property (see (4) below) can be seen as a generalization of the Path-independence axiom characterizing the choice functions studied in Danilov and Koshevoy (2005) and Plott (1973).

(3) Next we consider an intertemporal choice model where commodities are collected not only by their physical attributes, but also by the date at which they are bought and consumed. So, the DM's choice problem consists in tye selection of finite horizon consumption stream that is

the selection of commodities from sets of alternatives that are available at each time $t = 1$. With this interpretation in mind, we propose a third mechanism of choice from lists of sets of alternatives in which a DM *separately* chooses from each set their best alternative(s). It should be *rational* for her to focus on alternatives of great worth (*Matroidal axiom*, (**M**) see Danilov and Koshevoy, (2009)) and to neglect those ones that are valueless (Outcast axiom, (**O**)). We show that the DM's behavior who *separably* selects from each set of alternatives in a list and that is consistent with **M** and **O** rational principles is represented by a *dichotomous choice function* (see Danilov and Koshevoy, (2009)), namely that operator that divides alternatives in 'acceptable' and 'unacceptable' according to their value for the DM (see Theorem 3).

(4) Finally, we observe that if the elements of sets from a list are *not mutually exclusive*, the DM needs to look for a heuristic criterion that helps her decide the best solution-set of the many available to solve her choice problem. Namely, we need to introduce preferences over sets, i.e. *hyperrelation*, telling us when a set of alternatives A is at least as good as another set B. We show (see Theorem 4 below) that the **CFL** (see (18) below) that is induced by a hyperrelation fulfilling some compelling properties, satisfies **NR** if and only if the hyperrelation is generated by a choice function that complies with the Heritage rationality property.

The rest of the paper is organized as follows. Section 2 provides notation and the main definitions and reviews some elements of the classical choice model useful for comparison purposes. Section 3 and 4 show the main results, Section 5 contains some final comments, and all the proofs are in the last Section.

2 Choice functions on lists of sets of alternatives and the No-Regret property

Let X be a finite set of alternatives and 2^X be the set of all possible subsets of X. A *choice function* $f : 2^X \to 2^X$ is a contraction operator, i.e. for any $A \in 2^X$, $f(A) \subseteq A$. The set of all choice functions is denoted with $\mathbf{CF}(X)$. The empty choice is allowed, namely for some set $A \in 2^X$, it may be that $f(A) = \emptyset$.

A *list* $\mathcal{A}$ is a collection of non-intersecting sets $(A_1, \ldots, A_k)$ of alternatives of $A \in 2^X$, (i.e. $A_i \cap A_j = \emptyset$ for $i \neq j$), where $s := l(\mathcal{A})$ is the length and $\cup A_i$ is the *support* of the list $\mathcal{A}$. The set of all lists with support in all subsets of X is denoted by $\mathcal{L}$. Then, we can observe two kinds of choice from a list, analytically defined as follows:

Definition 1 *A mapping*

$$F : \mathcal{L} \to 2^X, \tag{1}$$

from a list $\mathcal{A} = (A_1, \ldots, A_k) \in \mathcal{L}$ into a subset of its support, i.e. $F(\mathcal{A}) \subseteq \cup_i A_i$, and a mapping

$$\mathcal{F} : \mathcal{L} \to \mathcal{L} \tag{2}$$

from a list $\mathcal{A} = (A_1, \ldots, A_k) \in \mathcal{L}$ into a sublist, i.e. $\mathcal{F}(\mathcal{A}) = (A'_1, \ldots, A'_k), with\, A'_t \subset A_t \ for\ t = 1, \ldots, k$.

We call both F and $\mathcal{F}$ *Choice Functions on Lists of sets of alternatives* (**CFL**).

Remark 1. We observe that we can get a **CFL** of type (2) from one of type (1), namely:

$$\mathcal{F}_F(\mathcal{A}) = (A_1 \cap F(\mathcal{A}), \ldots, A_k \cap F(\mathcal{A}))$$

and viceversa a choice function of type (1) from a $\mathcal{F}(\mathcal{A}) = (A'_1, \ldots, A'_k)$, where $A'_t \subset A_t$, with $t = 1, \ldots, k$, namely:

$$F_{\mathcal{F}}(\mathcal{A}) = \cup_t A_t^{'} \subset \cup_t A_t \quad for \quad t = 1, \ldots, k.$$

□

An explanation of **CFL** of the form (1) and (2), considers, for instance, that if A_t is the set of assets for trade at day t, with $t = 1, \ldots, k$, then the behavior of a DM, that trades *different* securities each day, can be represented by a choice function of the form (2), while, if A_t is a set of assets available at each time t of a day, and the choice is a set of securities at the end of a trading day, then DM's behavior can be described by a choice function of the form (1).
In what follows, we analyze the *rationality* of a DM whose behavior is represented by an element of the class of **CFL**. In other words, we study which choice function on lists of sets of alternatives might be considered *rational*. We usually regard as rational a choice function that satisfies to some extent some *principle of consistency*, namely some suitable criterion that, if satisfies, prevents any logical contradiction. In the classical choice model, where a DM chooses (one or some) alternatives from a set, she is considered to be *rational* if the function $f \in \mathbf{CF}(X)$, which describes her behavior, satisfies at least some of the following well-known *rationality* conditions, all much discussed in the theory of choice literature:

Heritage (**H**) For any A, $B \in 2^X$, if $A \subseteq B$ then $f(B) \cap A \subseteq f(A)$.

The Heritage condition (see e.g. postulate 4 in Chernoff (1954), Aizerman and Aleskerov (1995), Aizerman and Malishevski (1981)) means that if an alternative a is chosen from a set B, then it is also chosen from the smaller set $A \subseteq B$ including a. This is a very basic requirement in the classical approach to choice theory (see e.g. Moulin, (1985)).

Outcast (**O**) For any $A, B \in P(X)$, if $f(A) \subset B \subset A$, then $f(B) = f(A)$.

Property **O** (see e.g. postulate 5 in Chernoff (1954), Aizerman and Aleskerov (1995), Aizerman and Malishevski (1981), Danilov (2012)) says that removing the alternatives that are not chosen from a set does not affect the worth of the set.
We recall here that for single-valued choice functions, the Outcast and Heritage axioms are equivalent and provide a rational choice with respect to a linear order defined on the set of alternatives.

Plott (1973) proposed to consider a choice as *rational* if it does not depend on the way we divide the set of alternatives, namely the DM's choice does not depend on any particular order of presentation of the alternatives. This means that if a set A is divided into two subsets B and C, then making the choice from A must be the same as making a choice first from B and then from C, or the other way round, and finally making a choice from the union of these two choice sets. Analytically:

Path-independence (**PI**) For any $A \in 2^X$, with $A = B \cup C$,

$$f(A) = f(f(B) \cup f(C)). \tag{3}$$

A choice function that satisfies (3) is called a Plott function in Danilov and Koshevoy (2005) and usually *path-independent.*[1] We denote the set of all Heritage, Outcast and Path-independent choice functions with **He**, **Ou** and **Pl**, respectively. We then recall that for a choice function $f \in \mathbf{CF}(X)$ satisfying both **H** and **O** axioms is tantamount to satisfy path-independent property (see Aizerman and Malishevski (1981) and Lemma 6 in Moulin (1985)).

All the aforementioned axioms guarantee a certain *rational choice behavior.* In what follows, we introduce a *new rationality property*, that is a generalization (to the present more comprehensive choice model) of the Outcast property just reviewed above and therefore as such it has a strong rationality content.

We consider a DM as *rational* if she does not reconsider rejected alternatives when she makes a choice from lists of sets of alternatives: the alternatives that she disregards will never more be chosen. This rationality principle is what we propose here and call *No-Regret postulate* and it can be formalized as follows:

Definition 2 *(**No-Regret (NR) choice function**) A **CFL** $F : \mathcal{L} \to 2^X$ satisfies No-Regret property (**NR**) if, for any list $\mathcal{A} = (A_1, \ldots, A_k) \in \mathcal{L}$ and each list $\mathcal{B} = (F(\mathcal{A}) \cup B_1, B_2, \ldots, B_k) \in \mathcal{L}$ such that $\cup_j B_j \subseteq \cup_i A_i \setminus F(\mathcal{A})$:*

$$F(\mathcal{B}) \cap (\cup B_i) = \emptyset. \tag{4}$$

*Correspondingly, a $\mathcal{F} : \mathcal{L} \to \mathcal{L}$ is a **CFL** that satisfies No-Regret-property if for any list $\mathcal{A} = (A_1, \ldots, A_s) \in \mathcal{L}$, $\mathcal{F}(\mathcal{A}) = (A'_1, \cdots, A'_k)$, with $A'_t \subset A_t$, for $t = 1, \ldots k$, and for any $\mathcal{B} = (\cup_t A'_t \cup B_1, B_2, \cdots, B_l)$, $B_s \subset \cup_t (A_t \setminus A'_t), s = 2, \ldots, l$,*

$$\mathcal{F}(\mathcal{B}) = (A'', \emptyset, \cdots, \emptyset), \quad \textit{with} \quad A'' \subset \cup_t A'_t \tag{5}$$

The **NR** property means that if a DM makes a choice $F(\mathcal{A})$ (respectively, $\mathcal{F}(\mathcal{A})$) from a list $\mathcal{A} = (A_1, \ldots, A_k) \in \mathcal{L}$, then, for any list, with $F(\mathcal{A})$ as the first set and all other sets taken from alternatives that have not been chosen before, her choice will be from $F(\mathcal{A})$. A DM is considered to be *rational* if she does *not reconsider* rejected alternatives when she makes a choice from lists of sets. We further notice that $F(\mathcal{A})$ could be empty. In such a case, if a choice from some list $\mathcal{A}$ is empty, then a choice from any list, the support of which is a subset of the support of $\mathcal{A}$, is also empty.

The **NR** property is a variant of a sort of principle of *independence of irrelevant alternatives.* In particular, it is a generalization of the Outcast postulate equivalently defined as:

$$for\ any \quad B \subset A \setminus f(A), \quad f(B \cup f(A)) \subset f(A). \tag{6}$$

and exactly meaning that a DM does not select any previously not-considered alternative.

[1]The difference between these two classes of choice functions is that the non-emptiness of the choice sets is not required for the Plott functions.

Remark 2. We recall that:

(i) the class of choice functions satisfying the Outcast property is studied in Aizerman and Malishevski (1981), Brandt and Harrenstein (2011), and characterized in Danilov (2012);

(ii) **Ou** is stable under union, that is for any two choice functions $f, g \in \mathbf{CF}(X)$ satisfying (6), $f \cup g$, defined as $(f \cup g)(A) = f(A) \cup g(A)$, also satisfies Outcast axiom;

(iii) **Ou** is not stable under intersection and

(iv) any choice function might be obtained as intersection of Outcast choice functions (see Aizerman and Malishevski (1981)).

□

Remark 3 In order to familiarize the reader with **Ou**, we provide here the following two examples of choice function satisfying **O**:

(i) Let $B \subset X$ be a set of "bliss" elements of X, then the choice function $f_B(A) = B \cap A$, selecting the bliss-alternatives that, if available, are in A, is an Outcast choice function.

(ii) Let $\preceq$ be a weak order on X, that is a reflexive, complete and transitive binary relation, then $f_{\preceq}(A)$, that selects the set of maximal elements in A with respect to $\preceq$, is an Outcast choice function.

□

Before analyzing our new class of choice functions and its related rationality property, we first observe that, if F, G, (respectively $\mathcal{F}, \mathcal{G}$), are **CFL** satisfying **NR** then their union, defined as $(F \cup G)(\mathcal{A}) = F(\mathcal{A}) \cup G(\mathcal{A})$ is also a **CFL** that satisfies **NR**, i.e., in general:

Proposition 1 *The set of choice functions on lists of sets of alternatives that satisfies **NR** is stable under union.*

Proof Let $\mathcal{F}$ and $\mathcal{G}$ satisfy NR-property. Then we have to check that for any list $\mathcal{A} = (A_1, \cdots, A_k)$, and for any list $\mathcal{B} = (B_1, \cdots, B_k)$ such that

$$\cup_t B_t \subset \cup_t A_t \setminus \cup_t (A'_t \cup A''_t),$$

where $\mathcal{F}(\mathcal{A}) = (A'_1, \ldots, A'_k)$, $\mathcal{G}(\mathcal{A}) = (A''_1, \ldots, A''_k)$, there holds

$$(\mathcal{F} \cup \mathcal{G})(\mathcal{F}(\mathcal{A}) \cup G(\mathcal{A}) \cup \mathcal{B}) \cap \mathcal{B} = (\emptyset, \cdots, \emptyset) \tag{7}$$

Because of NR-property for $\mathcal{F}$ and $\mathcal{G}$, we have

$$\mathcal{F}(\mathcal{F}(\mathcal{A}) \cup G(\mathcal{A}) \cup \mathcal{B}) \cap \mathcal{B} = (\emptyset, \cdots, \emptyset), \tag{8}$$

and

$$\mathcal{G}(\mathcal{F}(\mathcal{A}) \cup G(\mathcal{A}) \cup \mathcal{B}) \cap \mathcal{B} = (\emptyset, \cdots, \emptyset) \tag{9}$$

From (8) and (9) we get (7).

□

We further notice that example (i) of Remark 3 above can be generalized to the present setting. Namely, for any set of *bliss* elements $B \subset X$, the **CFL**:

$$F_B(\mathcal{A}) = (B \cap A_1, \dots, B \cap A_k)$$

satisfies the **NR**-property. While, we also remark that, for a $f \in \mathbf{Ou}$, the **CFL**

$$\mathcal{F}(\mathcal{A}) = (f(A_1), \dots, f(A_k))$$

does not necessarily satisfy **NR** axiom.
Finally, we observe that the class of **CFL** satisfying **NR** has a maximal element, defined as:

$$\mathbf{1}(\mathcal{A}) = \mathcal{A}.$$

Therefore, it is a *semi-lattice* with respect to the union operator, exactly like **Ou**.

3 CFLs relying on choice functions

A DM who chooses from lists of sets and faces one set after another plausibly uses, as in the classical choice model, a choice function $f \in \mathbf{CF}(X)$ to directly make selection from a set of alternatives at a time. It is the case, for instance, of the choice from a list of a single set, namely $F(A) = f(A)$.

3.1 Iterative search

For a list $\mathcal{A} = (A_1, \cdots, A_k)$, we define the following *iterative* choice procedure:

$$\begin{aligned} C_1 &= f(A_1), \\ C_2 &= f(C_1 \cup A_2), \\ &\dots \\ C_k &= f(C_{k-1} \cup A_k). \end{aligned} \tag{10}$$

and set:

$$F^1_f(\mathcal{A}) = \cup_{i=1,\dots,k} C_i.$$

This means that, for example, if a list is composed by only two sets (A_1, A_2), we have that:

$$F^1_f(A_1, A_2) = f(A_1) \cup f(f(A_1) \cup A_2).$$

In words, a DM faces a set A_1 at time 1 and makes a choice $f(A_1)$, then she faces the set A_2 at time 2, but she already made the choice $f(A_1)$, so, at time 2, she has to choose from the set $A_2 \cup f(A_1)$ and a possible choice is a subset of A_2 and some subset of $f(A_1)$. However, since the choice $f(A_1)$ has already been made at time 1, the DM has to add $f(A_1)$ to the choice at time 2. The present choice mechanism describes how a search procedure of the desire alternatives *iteratively* depends on the time in which the DM faces them, but also by what she selected in the past. In other words, F^1_f reflects the dependence, for instance, of a choice at time 2 from that at time 1 and therefore how much a choice mechanism could be strictly connected over time.

It is worth noticing that the present choice procedure crucially *depends on the order of the sets in the list.* Indeed, for the list $\mathcal{A} = (A_2, A_1)$, we get $F_f^1(A_2, A_1) = f(A_2) \cup f(f(A_2) \cup A_1)$, that represents a choice set different from $F_f^1(A_1, A_2) = f(A_1) \cup f(A_2 \cup f(A_1))$: the present iterative choice mechanism is not *commutative*.

We suggest as a suitable interpretation for such a choice procedure to think of a college (the DM) that chooses from sets of candidates partitioned into groups according to their preferences for different colleges. Candidates who listed the college as the tth in their preference order appear at time t. Hence, the choice of a college from a list of candidates goes as follows: C_1 is the choice from the set of candidates who ranked this college as their first choice, C_2 is the choice from the union of C_1 and the set of candidates who were not chosen from the colleges they ranked first and ranked this college as their second option, and so on. At each next step, a college faces and chooses those who were rejected by other colleges and eventually add them to the previously chosen candidates for a new choice. We observe that in such a choice procedure, a college *can not reject candidates chosen at each previous step.*

Then, for F_f^1, a list $\mathcal{A} = (A_1, \ldots, A_k)$, the sets C_i defined as in (10) and any set $B \subset \cup A_i \setminus \cup C_i$, the **NR** axiom entails that:

$$f(f(C_1 \cup \cdots \cup C_k) \cup B) \cap B = \emptyset.$$

We are now ready to state under which conditions a **CFL** F_f^1 satisfying **NR** property could be considered as a representation of the behavior of a *rational* DM who chooses from lists of sets of alternatives following the choice mechanism under (10), namely:

Theorem 1 *For $f \in \boldsymbol{CF}(X)$, F_f^1 is a **CFL** that satisfies **NR** axiom if and only if $f \in \boldsymbol{Pl}$.*

Proof Let f be path independent. Then,

$$f(C_1 \cup \cdots \cup C_k) = C_k.$$

For a two step list $(f(C_1 \cup \cdots \cup C_k), D)$, we get

$$C_1' = C_k, \;\; C_2' = f(C_k \cup D).$$

Because of path independence, we have

$$f(A_1 \cup A_2 \cup \cdots \cup A_k) = f(f(A_1) \cup A_2 \cup \cdots \cup A_k) = f(f(f(A_1) \cup A_2) \cup \cdots \cup A_k) = \cdots = C_k.$$

Because of that $C_2 = f(f(A_1 \cup \cdots \cup A_k) \cup D) = f(A_1 \cup \cdots \cup A_k \cup D) = f(A_1 \cup \cdots \cup A_k) = C_k$. That is NR axiom is verified.

For the other direction. Let F_f satisfies **NR**. Then f satisfies the Outcast axiom and is idempotent. Now we have to check that the Heritage axiom is also satisfied. Suppose not that **H** is violated for a pair $A \subset B$. This means that $A \cap f(B)$ is not a subset of $f(A)$. Let us denote $C := (A \cap f(B)) \setminus f(A)$, C is nonempty. Consider the following partition

$$B = A \coprod B \setminus A.$$

Denote by $D := f(f(A) \cup (B \setminus A))$. Then, since C is non-empty, $E := f(B) \setminus D$ is also non-empty.

Because $f(B) \subset D \cup E \subset B$, due to **O**, we have

$$f(D \cup E) = f(B) \Rightarrow E = f(B) \cap E.$$

That is not the case due to NR-property, and, hence, the implication holds true.

□

Namely, the DM's choice from a list of sets of alternatives is *rational* if it does not depend on any particular order of presentation of the sets of alternatives, i.e. it is path-independent.
We finally notice that, for a path-independent f, the **CFL**:

$$\mathcal{F}_f(\mathcal{A}) = (A_1 \cap C_1, A_2 \cap C_2, \cdots, A_k \cap C_k)$$

does not have the form:

$$(f(A_1), f(A_2), \ldots, f(A_k)).$$

We study such a class of choice functions in Section 3.3.

3.2 Sequential search

For a list $\mathcal{A} = (A_1, \ldots, A_k) \in \mathcal{L}$, we consider the **CFL**:

$$G_f(\mathcal{A}) = C_k,$$

where C_k is defined as in (10), and $f \in \mathbf{CF}(X)$ is a choice function representing the DM's behavior. Namely:

$$G_f(\mathcal{A}) := f(A_k \cup (f(A_{k-1} \cup (\ldots \cup f(A_2 \cup f(A_1)))))) \tag{11}$$

G_f could be interpreted as a **CFL** representing the behavior of, for instance, a firm (hospital) that sequentially chooses from sets of applicants (doctors), who are divided according to their own ability (specialization), those who best fit the different vacancies it offers (see e.g. Chambers and Yenmez, (2017)). In a college admission model, F_f^2 stands for the selection mechanism from sets of candidates ordered according to their preferences for the different colleges. So, a college first chooses from the set A_1 to which belong all candidates who put the college as their best choice. Then, from the set of all candidates who indicates the college as their second choice and so on, with the *possibility for the college to reject previously chosen candidates*. Such a *sequential choice procedure* was analyzed in depth by Manzini and Mariotti (2007, 2012) and it is very much used for establishing the existence of *stable matching* (see e.g. Aygun and Sonmez (2012), Echenique and Oviedo, (2006) and Roth and Sotomayor (1990)).[2]

For this sub-class of **CFL**, the *No-Regret* axiom says that for any $f \in \mathbf{CF}(X)$, any list $\mathcal{A} = (A_1, \cdots, A_k)$, and any $B \subset \cup_i A_i \setminus C_k$:

$$G_f(G_f(\mathcal{A}) \cup B) \cap B = \emptyset. \tag{12}$$

[2]We recall that Fleiner (2003) showed that choice rule (11) fulfils *substitutability*, an essential property for the existence of a stable solution in matching market models (see Roth and Sotomayor, 1990). Koshevoy (1999) showed that *substitutability* property is equivalent to path-independent axiom.

The choice function (11) also shows the strategy of a DM who has to solve a sequence of choice assignments, which consists in storing the solution(s) to the first assignment solved and in solving the next in the list. At the end, she builds on all the solutions she found to the different sub-assignments to reach her final goal. A necessary condition to succeed in coping with the complexity of her choice tasks is that the sequential choice protocol disregard alternatives they are somehow of no value to the DM.

We show that the class of **CFL** in (11) that satisfies **NR** in (12) coincides with the class of choice functions (closed by set-union) introduced by Plott (1973), namely:

Theorem 2 *For any $f \in \boldsymbol{CF}(X)$ and any $\mathcal{A}= (A_1, \dots, A_s) \in \mathcal{L}$, the following statements are equivalent:*

1. *G_f satisfies **NR**-property in (12);*
2. *$f \in \boldsymbol{CF}(X)$ is path independent.*

Proof The proof follows the same line of arguments of the proof of Theorem 1 and therefore it is omitted.

The rationality of whom choosing *sequentially* from sets of alternatives in a list, disregards alternatives that do not help solve her choice problem, coincides with the (pseudo-)rationality (see Moulin, 1985) of a DM whose choice is not influenced by the particular position of an alternative in a list. It is worth observing that a DM who adopts the sequential choice procedure under (11) is 'immune' to *manipulation* of the alternatives by putting them in a particular order (*strategy-proof*). Finally, and importantly, Theorem 2 can be considered as a *new characterization* of the rich class of choice functions proposed by Plott (1973) and therefore *No-Regret* property in (4) can be read as an extension of the path-independent rationality axiom to the present more general setup. Finally, we observe that $\mathcal{F}_f^2$ is invariant under permutations of the sets of the list, while $\mathcal{F}_f$ is not.

3.3 Separable choice

Another interesting case of **CFL**, built by applying a choice function $f \in \mathbf{CF}(X)$ to each set in a list, is the following:

$$\begin{aligned} C_1 &= f(A_1), \\ C_2 &= C_1 \cup f(A_2), \\ &\dots \\ C_k &= C_{k-1} \cup f(A_k). \end{aligned} \tag{13}$$

with:

$$I_f(\mathcal{A}) = C_k = f(A_1) \cup f(A_2) \cup \dots \cup f(A_k).^3$$

At least three possible suggested interpretations for the choice procedure under (4) help the reader to better understand the underlying selection mechanism, Indeed, I_f could represent the

[3] We notice that the corresponding **CFL** of type (2) takes here the form, $\mathcal{F}_f(\mathcal{A}) = (f(A_1), \dots, f(A_k))$.

behavior of a DM who fills her wish-list choosing from (a list of) different departments (the sets of alternatives) in an online market. Still, she chooses her meal selecting from the appetizers, the desserts etc, i.e. the sets of different dishes in which a menu (the list) is usually divided. Finally, she buys different bonds from the sets A_t of securities available at each period of time t, with $t = 1, \dots k$. In general, a DM chooses the best elements from the sets of alternatives available at each certain period, independently on the choices she made in other moments.

In the present case, the **NR**-property reads as follows:

$$f(A \cup B) \subset f(A) \cup f(B), \tag{14}$$

$$f(A \setminus f(A)) = \emptyset, \tag{15}$$

and

$$for\ any \quad B \subset A \setminus f(A), \quad f(B \cup f(A)) \subset f(A) \tag{16}$$

where (14) says that the choice from the union of two sets is a smaller set than the set obtained as the union of the choice of the two subsets; (15) is the *Matroidal* property identified in Danilov and Koshevoy (2009) and meaning that deleting a 'good' alternative does not make 'bad' alternatives 'good'.[4] and (16) is a restatement of the *Outcast* property. In order to characterize F_f^3, we need to introduce the notion of *dichotomous* choice function, namely those $f \in \mathbf{CF}(X)$ such that, for any $A \in 2^X$, $f(A) = A \cap f(X)$, i.e. those choice functions that divide all alternatives into "*acceptable*" (those belonging to $f(X)$) and *non-acceptable*. Thus, for any $A \in 2^X$, a dichotomous choice function f only selects all the acceptable alternatives in A.[5]
Thus, we state that:

Theorem 3 *For a choice function $f \in$ **CF**(X), I_f is a **CFL** that satisfies the **NR** property if and only if, for some $B \subset X$, f_B is dichotomous.*

Proof It is easy to check that, for a dichotomous f_B, I_{f_B} satisfies **NR**-property. Vice versa, from **NR** sub-(14), (15) and (16), we obtain, for $x \in f(X)$, that $f(x) \neq \emptyset$, and, for any $y \in X \setminus f(X)$, that $f(y) = \emptyset$. Then, we have:

$$f(A) = A \cap f(X).$$

In fact, if $f(A) = \emptyset$, then this obviously holds. Let $f(A) \neq \emptyset$, then, due to (14), for any $a \in f(A)$, $f(a) \neq \emptyset$ and hence $a \in f(X)$.

□

4 CFLs induced by hyperrelations

A DM who chooses from lists of sets of alternatives needs to make comparisons between the elements of a set and the elements of the set next in a list in order to find alternatives that are

[4]We also observe that, according to the *Matroidal* property, deleting some 'bad' alternative can transform some other 'bad' alternative into 'good' one.

[5]We recall that a dichotomous choice function satisfies both *Heritage* and *Matroidal* property (see Danilov and Koshevoy, (2009).

solutions to her choice problem. However, if the members of the sets to compare are not mutually exclusive as for instance in matching theory (see e.g Echenique and Oviedo (2006) and Roth and Sotomayor (1990)), in certain voting procedures (see e.g. Brams and Fishburn, (2002)), in coalition formation (see e.g. Ray and Vohra, (2014)) and in sequentially rationalizable choices (see e.g. Manzini and Mariotti (2007) and (2012)), it may be necessary to take *preferences on sets*, i.e. binary *hyperrelations* (see e.g. Danilov, Koshevoy and Savaglio, 2015), as a primitive.

Let us consider the following very compelling properties that a hyperrelation $\prec$ should satisfy (see Danilov, Koshevoy and Savaglio, (2015)):

- **Monotonicity with respect to set inclusion (Mon).** For all $A, B \in 2^X$, $A \subseteq B$ implies $A \prec B$.
- **Stability with respect to contraction (Cont)**. For any A, A', $B \in 2^X$, $A' \subseteq A \prec B$ implies $A' \prec B$.
- **Stability with respect to extension (Ext)**. For any A, B, $B' \in 2^X$, $A \prec B \subseteq B'$ implies $A \prec B'$.
- **Union (U)** For any A, A', $B \in 2^X$ $A \prec B$ and $A' \prec B$ imply $A \cup A' \prec B$.

Mon is a suitable axiom much discussed in the economic literature on ranking sets of alternatives in terms of freedom of choice (see Barberà, Bossert and Pattanaik, 2004). **Cont** and **Ext** reflect the relationships between a hyperrelation and set-inclusion: if a set provides more suitable alternatives than another, then this *a fortiori* holds for any subset of the latter (i.e. **Cont**), and if a set offers more 'suitable alternatives' than another set, then the set containing the former as its subset will certainly provide more 'suitable alternatives' than the latter (i.e. **Ext**). **Cont** and **Ext** may be considered as a weakening of the transitivity property. Namely, if a transitive hyper-relation $\prec$ satisfies **Mon** then $\prec$ is stable with respect to contraction and extension. The Union property simply says that the union of two sets both worse than another is still worse than the latter. We observe that the **U** axiom is tantamount to the Robustness property in Barberà, Bossert and Pattanaik (2004).

A hyperrelation $\prec$ that satisfies **Mon**, **Cont**, **Ext**, and **U** is called *decent* in Danilov, Koshevoy and Savaglio (2015). They showed that all the aforementioned properties make a hyperrelation $\prec$ connected with the important class of choice functions satisfying Heritage property, a property that plays a crucial role in establishing the rationality of a choice. They indeed proved, for a choice function $f \in \mathbf{CF}(X)$ and any $A, B \subset X$, that the hyperrelation:

$$A \kappa_f B \quad \text{if, for every} \quad a \in A, \quad f(a \cup B) \subseteq B \tag{17}$$

is *decent* if f satisfies Heritage property and vice versa any hyperrelation that is *decent* takes the form of κ_f with f that is a Heritage choice function (see Theorem 1 and Proposition 2 in Danilov, Koshevoy and Savaglio, (2015)).

Another compelling property for a decent hyperrelation $\prec$, was intruduced by Puppe (1996), who called it 'Independence of Non-essential Alternatives'. It makes possible in a set to distinguish some alternatives, that could be considered as "essential' and some others that are somehow *worthless* for the DM's choice problem. Analytically,

- **Framed (F)** Let $\prec$ be a decent hyperrelation and f its corresponding choice function, then, for every $A \in 2^X$, $A \prec f(A)$.

In what follows, we investigate the relation between the foregoing axiomatic structure and the class of choice functions on lists of sets of alternatives satisfying **NR**. In order to do that, we consider a hyperrelation $\prec$ and the following **CFL**:

$$R_{\prec}(A_1, \ldots, A_k) = \cup_{j \in J} A_j, \tag{18}$$

where $j \in J$ if and only if there is no $j' < j$ such that $A_j \prec A_{j'}$.

In words, choice function (18) selects the top elements with respect to the hyperrelation $\prec$ among those of the collection $(A_1, \ldots, A_k)$. The choice mechanism proceeds as follows:

$$P_1 = A_1$$

$$P_2 = \begin{cases} P_1 \text{ if } P_1 \succ A_2 \\ A_2 \text{ if } P_1 \prec A_2 \\ P_1 \cup A_2 \text{ otherwise} \end{cases}$$

$\ldots \ldots$

$$P_k = \begin{cases} P_{k-1} \text{ if } P_{k-1} \succ A_k \\ A_k \text{ if } P_{k-1} \prec A_k \\ P_{k-1} \cup A_k \text{ otherwise} \end{cases}$$

and it defines:

$$R_{\preceq}(\mathcal{A}) = P_k.$$

This means that a DM adds A_t at time t if it is not dominated by the union of previously chosen sets.

Then, we show that:

Theorem 4 *If $\prec$ is a monotone hyperrelation satisfying the condition [$A \prec B$, $C \prec B$ implies $A \cup C \prec B \cup C$], then $R_{\prec}$ satisfies **NR** if and only if $\prec$ is decent and framed.*

Proof ($\Rightarrow$) We have to show that if $R_{\prec}$ satisfies **NR**, then the monotone hyperrelation $\prec$ satisfies **Cont**, **U** and **Ext** and is *Framed,*.
Let $\mathcal{A} = (A_1, A_2)$ be any two-sets partition of $A \in 2^X$, then:

$$R_{\prec}(A_1, A_2) = \begin{cases} A_1 \\ A_2 \\ A_1 \cup A_2 \end{cases}.$$

Cont follows from a direct application of (18) satisfying **NR** to $\mathcal{A} = (A_1, A_2)$.
Consider now the list $\mathcal{A} = (A_1, A_2, A_3)$ of $A \in 2^X$ and suppose that $A_1 \prec A_3$ and $A_2 \prec A_3$, then

by **NR** the fact that $A^{'} \prec A_3$ for any $A^{'} \subseteq A_1 \cup A_2$ entails that $A_1 \cup A_2 \prec A_3$, i.e. **U** is satisfied. Take a list $\mathcal{A} = (A_1, A_2 \setminus A_3, A_3)$ and suppose that $A_1 \prec A_3 \subset A_2$, then

$$R_{\prec}(A_1, A_2 \setminus A_3, A_3) = \begin{cases} A_3 \\ A_2 \end{cases} .$$

If $R_{\prec} = A_2$, we have that **Ext** is satisfied. If $R_{\prec} = A_3$, then $A_1 \cup A_2 \setminus A_3 \prec A_3$ and $A_2 \setminus A_3 \prec A_3$. By Theorem's assumption $A_1 \cup A_2 \setminus A_3 \prec A_2$ and by **Cont** $A_1 \prec A_2$, i.e. **Ext** is satisfied. Hence, $\prec$ is decent and by Proposition 2 in Danilov, Koshevoy and Savaglio (2015) we conclude that the $f \in \mathbf{CF}(X)$ inducing $\prec$ satisfies **H**. In particular, it follows, by applying **NR**, that $\prec$ is also *framed*. Therefore, by Theorem 3 in Danilov, Koshevoy and Savaglio (2015) $f \in \mathbf{CF}(X)$ is also *Outcast* and then, by Aizerman and Malishevski (1981), $f \in \mathbf{CF}(X)$ satisfies *path-independence*.

($\Leftarrow$) Let $\prec$ be a decent hyperrelation, then it satisfies **Ext** and by Proposition 2 in Danilov, Koshevoy and Savaglio (2015) it must be induced by a $f \in \mathbf{CF}(X)$ that satisfies **H** and takes the form of $\prec_f$. Moreover, $\prec$ is also *framed* and therefore, by Theorem 3 in Danilov, Koshevoy and Savaglio (2015), f must satisfy *Outcast*. We then have to show that the corresponding choice function on lists of sets of alternatives $R_{\prec_f}$ satisfies **NR**. Suppose by contradiction that $R_{\prec_f}$ does not satisfy **NR**. We know that $A_j \in R_{\prec_f}(A_1 \cdots, A_k)$ means that $A_i \prec A_j$ for any $i \neq j$, i.e. $f(a \cup A_j) \subset A_j$ for any $a \in A_i$. Suppose that for some entry a of the union of the sets that are not in $R_{\prec_f}(A_1 \cdots, A_k)$ we have that $a \in f(a \cup R_{\prec_f}(A_1, \cdots, A_k)$. Then, there exists a $A_t \in R_{\prec_f}(A_1, \ldots, A_k)$ such that $a \notin f(a \cup A_t)$ that contradicts **H**. Again, suppose that **NR** doesn't hold true. This means that for any $B \subset \cup A_i \setminus R_{\prec_f}(A_1 \cdots, A_k)$ there exists some $a \in B$ such that $a \in f(a \cup R_{\prec_f}(A_1, \cdots, A_k)$ and $a \notin R_{\prec_f}(A_1, \cdots, A_k)$, that contradicts (16), i.e. *Outcast*. Hence, $R_{\prec_f}$ must satisfy **NR**.

□

Because of Proposition 2 and Theorem 3 in Danilov, Koshevoy and Savaglio (2015), Theorem 4 states the case in which *No-Regret* property is equivalent to *Path-independent* axiom. Moreover, Theorem 4 establishes that if the DM, obeying to **PI** postulate, is rational then she chooses from a list of sets of alternatives by collecting only those sets that are not dominated, namely the ones that contain alternatives valuable to her. On the other hand, if she disregards those elements (sets) of a list that are worthless for her because she will never choose any of their alternative, then her rationality behavior is exactly that of a DM who is '*pseudo-rational*' namely, her choice cannot be manipulated by the presentation of the sets of alternatives in a list.

5 Summary and relation with the literature

We extend the classical choice set-up, in which a DM chooses from a set of alternatives, to the case in which she makes her selection from *lists of sets of alternatives*. We collect some general procedures of choice from lists of sets of alternatives and study a corresponding *new rationality property*, showing that it is quite general to encompass and extend other rationality notions already discussed in the theory of choice literature.

Our proposal is definitively linked to Rubinstein and Salant (**RS**, (2006)), that analyzes a choice model in which the DM encounters alternatives in the form of *lists of singletons*. In that paper, the set of all possible lists of alternatives is a set of linear orders on X, namely the set of the permuted elements $(x_{\pi(1)}, \ldots, x_{\pi(n)})$ of $X = \{x_1, \ldots, x_n\}$, where $\pi : \mathtt{N} \to \mathtt{N}$ is a permutation function. For this special case of lists of singleton sets, our rationality property of No-Regret requires that if x_i is the choice from a list, then it will be the choice from any list that starts with x_i.

RS (2006) shows that a choice function f on a list of alternatives $\mathcal{A} = (a_1, \ldots, a_n)$ satisfies the Path-independent property if and only if f satisfies Outcast axiom. Namely, a DM chooses one alternative from a list or equivalently that same alternative contained in any of the possible disjointed sublists in which the list could be split (*pseuforationality*) and this is consistent with choosing that alternative neglecting all others as useless for her (*Outcast* rationality). In Our setting, **RS** (2006) can be seen as a model of choice from sublists of alternatives as the following example shows. Let M be a set of 'top elements' of X, the choice function on lists of alternatives:

$$F_M(x_1, \ldots, x_k) = x_j$$

if and only if

$$x_j = \{x_1, \ldots, x_k\} \cap M, and \ \{x_1, \ldots, x_{k-1}\} \cap M = \emptyset.$$

satisfies the No-Regret property, that in such specific case is equivalent to the Outcast axiom, i.e. a DM chooses x_k if and only if it is the earliest occurrence of the top elements in the (sub-)list $(x_{i_1}, \ldots)$. Moreover, we remark that **NR** axiom, for such a case, is equivalent to Outcast property and, by Proposition 1 in Rubinstein and Salant (2006), to Path-independent axiom making clear how **NR** puts into relation choice from lists of sets with choice from sub-lists.
We conclude mentioning, among others, the following prominent papers analyzing choice from lists of alternatives of Guney (2014), Dimitrov, Mukhererjee and Mutu (2016), Yildiz (2016), Masatlioglu and Nakajima (2013), Ishii, Kovach and Ulku (2021). All these works propose quite complete guidelines for choice protocols from lists, which invariably differ from our own proposal in some significant respects. They study the case of choice from lists of alternatives as opposed to our more general case of lists of sets of alternatives. They provide a rational to some (real-life met) selection procedure of a single best alternatives out of a list, while we study the *rationality* of a DM who makes selections by using general choice mechanisms from lists of sets of alternatives, a definitively different exercise.

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