# Optimal switching from cooperation to competition: a preliminary exploration

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**Abstract**

Existing literature considers more often merging or collation under different circumstances. Few efforts were made to investigate the optimal condition of separation. This study is trying to fill in this gap. Furthermore, many economic and operational research studies have analysed the optimal timing of switching between different regimes. Most of these studies focus on the situation of the same decision maker in different regimes. In this paper, via differential game, we first notice that the classical multiple stage optimal regime switching condition obtained via Maximum principle is no longer working due to the decision makers changed before and after the separation. Thus, there may be different choices because of shortage of transversality condition between different periods. Dynamic programming does not depend on this kind of trasnversality condition, and can provide the optimal time of separation from an international agreement or treaty. Nonetheless, the optimal switching time depends essentially on the choice of strategic space after separation.

**Keywords:** Strategy spaces; cooperation and competition; differential games; multi-stage optimal control.
**JEL classification:** C61, C73, D71.

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# 1 Introduction

The recent years have noticed numerous withdrawals of countries from international organizations: (1) the most recent happens on July 7, 2020, the Trump administration formally notified the United Nations that it is pulling out of the World Health Organization, which effective as of July 6th, 2021;[1] (2) during the same Trump presidency period, on June 1, 2017, President Trump announced that the U.S. would cease all participation in the 2015 Paris Agreement on climate change mitigation until some fair conditions to the U.S.A can be negotiated; (3) the United Kingdom withdrew from the European Union on January 31, 2020; (4) Canada withdrew from Kyoto Protocol on December 13, 2011... Except political separation and withdrawals, companies' splitting (and merging) happen all the time. Some of the splitting (and merging) also have quite influential social and market effects, such as, October 2014, technology giant Hewlett-Packard, known as HP, splits itself into two separate companies: separating its computer and printer businesses from its faster-growing corporate hardware and services operations, and eliminate another 5,000 jobs as part of its turnaround plan; July 2015, PayPal spinoff from eBay and this split benefits both eBay's marketplace business by letting it accept different forms of electronic payment and also gives PayPal more autonomy to work with other potential partners, such as Amazon or Alibaba...

Obviously, this kind of phenomenons drew lots of attention in economic literature. There are quite some economic papers investigating the impacts of Brexit( Sampson, 2017; Latorre et al., 2020; the special issue of Oxford Review of Economic Policy, vol 33, 2017; etc.), U.S. withdrawal from the environmental related Kyoto Protocol and Paris agreement (Bucher et al., 2002; Dai et al., 2017; Nong and Siriwardana, 2018; ...), empirical study of Mayer et al (2019) on the cost of being non-EU, and the general theoretical investigation of Gancia et al (2020) on the gain of being in some economic unions and partnerships. Nevertheless, all these studies are silence about the timing of switching from cooperation to non-cooperative competition and ignore the possible choice of strategic space while separating happens.

The above list splitting phenomenons do not just happen randomly, rather the timing is well-planned and strategies are carefully chosen. Notwithstanding, as clearly stated

[1] But at the same time, Joe Biden, who will challenge Donald Trump in the November 2020 presidential election, tweeted: "On my first day as President, I will rejoin the WHO and restore our leadership on the world stage."

by Boucekkine et al (2020), in the economic and operational research literatures, the optimal timing of transitions (if any) are only implicitly tackled though the vast majority of the models developed are dynamic.[2] There exists however an increasing number of papers interested in optimal regime transition and the inherent timing (Tomiyama, 1984; Boucekkine et al., 2013; Boucekkine et al., 2004; Moser et al., 2014; Saglam, 2011; Zampolli et al. 2016; etc.). Commonly, all these studies use multi-stage optimal control techniques. Furthermore, there are few economic literature merging multi-stage optimal control and dynamic games ingredients, though the corresponding operational research literature is less poor (see for example Boucekkine et al., 2011).

A quite rich set of questions arises from the examples given above: What are the tradeoffs involved in the decision to move from cooperation to competition (and vice versa), and what is the optimal timing for the institutional regime change if any? What are the conditions that the coalition is always better than separation and from whose point of view it is better?

In this paper, we propose a preliminary exploration of the switching problem mentioned above. To this end, we solve a two-stage optimal control problem. In the first stage, two players cooperate on a common state variable which could be a public good or a public bad. Then, when separation happens, the disunion players engage in a dynamic competition game in the second stage. Arguably, under the current setting, in the second stage, the essential interesting question is the equilibrium concepts: *path strategies*, i.e., open-loop strategies, with commitment when separation happens, or *decision rule strategies*, i.e., the Markovian strategies, where players can adjust their strategies with the process of the competition, or the mixture of the two. The games are solved via backward induction, thus, the choices of strategy spaces not only matter for the outcomes of the second stage, but also are crucial for the switching time and the choices of the first stage optimal control problem.

The paper is organized as follows. Section 2 briefly presents the differential game setting and clearly describes the possible choices of strategy spaces. Section 3 provides the optimal switching time under two different Makovian strategies: one is both players adopt Markovian strategy; and the other is only one player did so while the other one stays with the first best choice before separation. In Section 4, some comparison studies take places in terms of switching time, social welfare as well as long-run state situations. Section 5

[2]This is specially true in the political transitions literature, with the notable exception of Boucekkine et al., 2016.

concludes.

# 2 A simple model of optimal switching and strategic spaces

## 2.1 The model

Suppose there is one bloc or coalition, such as the Kyoto Protocol, the Paris climate Agreement, and that one of the bloc's players, named player $i$, would like to quit the bloc at some future date $T$. The rest of the bloc is named as player $J$. The two players, i and J, share a common variable, $y \in [0, Y]$.

When players act as one bloc, players i and J choose jointly the level of variables $x_i, x_J \in [0, X] \subset [0, +\infty)$, which provide them with a joint utility. Their choices for $x_i$ and $x_J$ increase the level of $y$, which induces a loss in utility. Let us assume that at time 0, players play cooperatively until time $T$, when player i decides to quit the bloc. Note that at time $T$, player $J$ may also switch her strategy.

Let us provide an economics example that will accompany us throughout the paper. Suppose there exists a unique final good, which requires only a polluting resource as input. With a quantity of pollution $x_j$, $j \in \{i, J\}$, player $j$ produces accordingly final good. The consumption of this final good provides players with utility, but at the same time it increases the level of $CO_2$ emissions, $y$. Obviously, the level of $CO_2$ affects both players. In the end, player $j$ can obtain utility directly from the consumption of $x_j$, but she also suffers from pollution, since $y$ brings disutility.

Initially, the objective of the players in the bloc is to maximize joint overall welfare, defined as

$$\max_{x_i, x_J} W(\infty) = \int_0^{+\infty} e^{-rt} \left[ u_i(x_i) + u_J(x_J) - c_i(y) - c_J(y) \right] dt, \tag{1}$$

where $r$ is the time discount rate, $u_i$ and $u_J$ are, respectively, utility functions of player i and J, which are strictly increasing and concave; and $c_i(y), c_J(y)$ are their respective disutility due to pollution, which are strictly increase and convex. Obviously, we assume that the jointed objective function is simply the sum of the two players' objectives. On the other hand, this is the simplest way to define the jointed utility and on the other hand it would allow us to focus on the effects of strategic choices instead of the difference from objective functions between jointed and individual players.

Decisions are subject to the dynamic constraint:

$$\dot{y}(t) = f(x_i, x_J, y) = x_i + x_J - \delta y(t), \quad y(0) = y_0 \text{ given,} \tag{2}$$

and $\delta \in [0, 1]$ is the depreciation rate. In our example, $y$ stands for $CO_2$ emissions, so that $\delta$ would stand for the natural reabsorption rate of $CO_2$ in the atmosphere.

Suppose that player i's share in the total welfare is $\alpha \in (0, 1)$ and the remaining share, $1 - \alpha$, belongs to the rest of the bloc, i.e., welfare of players i and J are

$$W_i = \alpha W(\infty) \text{ and } W_J = (1 - \alpha)W(\infty).$$

Noteworthy, the share $\alpha$ is independent of $x_i$. We assume that $\alpha$ is given at the initial date and that renegotiation is impossible or too costly [3]. Note that this could be one of the reasons why player i could decide to quit the bloc at some future date $T$ and play non-cooperatively. Another possible reason is that the commitment made at time $t = 0$ to the trajectory $x_i$ may seem unfair or too strict for player i at some later date.

If player $i$ quits the bloc at time $T$, then as already said, she obtains a share $\alpha$ of overall welfare until time $T$. From time $T$ onwards, player i's objective becomes

$$W_{i,II} = \max_{x_i} \int_T^{+\infty} e^{-rt} \left[u_i(x_i) - c_i(y)\right] dt, \tag{3}$$

and player J faces

$$W_{J,II} = \max_{x_J} \int_T^{+\infty} e^{-rt} \left[u_J(x_J) - c_J(y)\right] dt, \tag{4}$$

subject to the same state equation:

$$\dot{y}(t) = f(x_i, x_J, y) = x_i + x_J - \delta y(t), \quad t \geq T, \tag{5}$$

where initial condition $y(T)$ comes from the outcome of the first period.

[3] One way to avoid player i quitting the bloc is allowing renegotiation such that $\alpha$ is a function of contribution, $\alpha = \alpha(x_i)$, or ratio of contribution: $\alpha = \alpha(x_i, x_J)$.

The optimal switching time for player i is defined as

$$\max_T \left( \alpha W(T) + \int_T^{+\infty} e^{-rt} \left[ u_i(x_i) - c_i(y) \right] dt \right) = \max_T \left( \alpha W(T) + W_{i,II} \right), \tag{6}$$

where $W(T)$ is the same integral in (1) but over time interval $[0, T]$:

$$\int_0^T e^{-rt} \left[ u_i(x_i) + u_J(x_J) - c_i(y) - c_J(y) \right] dt.$$

Intuitively, the first term in (6) is non-decreasing in term of $T$, that is, the longer the time period, the higher is the social welfare, otherwise $T = 0$ already. Similarly, the second term in (6) is non-increasing with respect to $T$, otherwise $T = +\infty$. Obviously, the precise optimal choice of $T$ relys on the strategy space after the separation. Under some parameter setting, $T = 0$ is the optimal, i.e., no union or bloc at all; with some other parameter setting, $T = +\infty$ may be the optimal choice; and of course still with other setting the optimal $T$ checks $0 < T < +\infty$.

The interior optimal switching time $T$ should be given by solution of

$$\alpha \frac{dW(T)}{dT} + \frac{dW_{i,II}}{dT} = 0,$$

provided the second order optimality condition

$$\alpha \frac{d^2W(T)}{dT^2} + \frac{d^2W_{i,II}}{dT^2} < 0$$

holds.

In the discussion section, we give special attention to the situation where $T = +\infty$, that is the coalition will continue forever. Here, we first suppose specially, there exits unique interior solution: $0 < T < +\infty$, then implicit function theorem yields

$$\frac{\partial T}{\partial \alpha} = -\frac{\frac{dW(T)}{dT}}{\alpha \frac{d^2W(T)}{dT^2} + \frac{d^2W_{i,II}}{dT^2}} > 0.$$

In other words, if player i is the dominating player in the bloc. i.e., occupying a relatively larger share from the aggregate welfare, she quits the bloc later. And the small player would rather quit the bloc earlier to gain some freedom and free from commitments.

It is worth mentioning that time $T$ should not consider as truncated terminal time, rather it is the time the separation happens. Different from most of the optimal switching literature, under the current setting, before and after time $T$, players change: before time $T$, the bloc makes choice for the bloc as one while after $T$ there are two competing players. Thus special care should be taken when employing the usual necessary optimal switching conditions at $T$. This difficulties come mainly from the choice of different strategic spaces after the separation. In the following sections, we introduce some of these cases.

## 2.2 The choices of strategic spaces

If all players act as one bloc from time 0, then the solution to the bloc's optimal control problem forms the first best choice. When a player decides to leave the bloc at time $T$, the common strategy space for the game when $t \in [0, T]$ is no longer valid after $T$. As a result, it is necessary to decide the strategic spaces for both players i and J from time $T$ on. As mentioned by Reiganum and Stokey (1985, Page 162) " *when formulating a model care should be taken to choose a strategy space that is appropriate for the situation under study*". Under the current setting, the choice of strategic spaces is rather complicated and a few alternative scenarios raise.

In the first scenario both players i and J recalculate their optimal strategies depending on the state. This is the standard differential game with Markovian strategies starting from time $T$. One example of this situation is the Brexit, which describes the exit of the UK from the EU. Among others, the Brexit induces the UK and the EU to recalibrate their commitment to the Paris Climate Agreement. When the Agreement was signed in December 2015, the UK was part of the EU and had put in its effort as part of the EU. At that time, the EU presented the Intended National Determined Contribution (INDC) on behalf of all the 28 Members States. After the Brexit, the EU has to recalibrate its INDC and the UK has to present its own contribution. Of course, if UK wants to pursue stronger climate actions, the British government can potentially submit more ambitious INDC and decide to implement stronger climate policy accordingly.

In a second scenario, player J sticks to its original commitment while player i updates her own strategy depending on the state of pollution, that is, player i plays Markovian strategies. The United States withdrawal from the Paris Agreement in 2017 belongs to this strategy case (and rejoined in 2021 after new presidency). Obviously, the other countries from the Paris Agreement did not recalculate their strategies. The U.S., at least during

Trump's presidency, did not commit to new target either. [4] As in the second scenario, the problem becomes an optimal control problem for player i taking as given player J's commitment.

For example, a third scenario could be that only player i recalculates and commits her new optimal strategy, which may be different from her Markovian strategy. Player J does not recalculate hers, and she rather stays committed to the first best resolution. The Canadian withdrawal from the binding Kyoto Protocol in 2011, and rejoining non-binding agreement to the Copenhagen Accord in 2009 matches this strategy case. On the one hand, Canada did not manage to reach their initial committed target to Kyoto Protocol and faced billions of dollars penalties. On the other hand, Canada still believed in the need of reducing greenhouse gas emissions. As a result, Canada committed in 2009 to the non-binding Copenhagen Accord with a recalculated target. In the meantime the remaining countries of the Kyoto protocol (player J in our model), did not adjust their commitment. Of course, under the new situation, player J's commitment is no longer optimal, but commitment is commitment and it should be respected. Starting from $T$, the problem becomes an optimal control problem for player i alone. Thus, player i takes as given player J's time variant strategy, which usually makes the problem more difficult to solve mathematically.

A fourth setting could be that player i updates her strategy based on the state variable (i.e., player i adopts a Markovian behavior), and player J knows that. Instead of sticking to the first best commitment, player J recalculates and adopts a new commitment (thus open-loop behavior). Note that player J takes into account the fact that player i plays a Markovian strategy. This is the so called Heterogeneous Strategic Nash equilibrium a la Zou (2016). Again, due to the withdrawal of the US from the Paris Agreement, Dai et al (2017) demonstrate that: "*under the condition of constant global cumulative carbon emissions and a fixed burden-sharing scheme among countries, the failure of the U.S. to honor its National Determined Contribution commitment to different degrees will increase the U.S. carbon emission space and decrease its mitigation cost. However, the carbon emission space of other parties*", thus the remaining countries in the agreement, "*will be reduced and their mitigation costs will be increased*". Given that the U.S. plays Markovian feedback strategies and that it does not commit to any new target, the commitment of the

[4] Although the U.S. did not set any target as a country, the governors of several U.S. states formed the United States Climate Alliance to continue to advance the objectives of the Paris Agreement at the state level despite the federal withdrawal.

remaining bloc has to take that into account. Another example of this kind of scenario comes from the US withdrawal from Kyoto Protocol in 2001 and the remanning parties' Bonn Agreement.

Although there may be other cases, we focus here on the first two scenarios in more detail via linear-quadratic form as Jun and Vives (2004).

# 3 The linear-quadratic model

To illustrate more clearly the outcomes of the above model, in the rest of the paper, we take linear-quadratic functional forms. The utility functions are

$$u_i(x_i) = a_i x_i - \frac{x_i^2}{2}, \quad u_J(x_J) = a_J x_J - \frac{x_J^2}{2},$$

and pollution damage functions are

$$c_j(y) = \frac{by^2}{2}, \quad j = i, J.$$

In other words, regardless the development level, the pollution damage is the same for both players for simplicity. $x_j$ can be considered as pollution emission of player j in order to produce final consumption goods, $a_j$ is efficiency parameter which converts the pollution into the consumption good. Thus, higher $a_i$ indicates more advanced economies which can convert more consumption from the same unit of pollution.

The jointed welfare function is

$$\max_{x_i, x_J} W(\infty) = \int_0^{+\infty} e^{-rt} \left[ a_i x_i + a_J x_J - \frac{x_i^2 + x_J^2}{2} - by^2 \right] dt, \tag{7}$$

subject to the following dynamic constraint:

$$\dot{y}(t) = x_i + x_J - \delta y(t), \quad y(0) = y_0 \text{ given}, \tag{8}$$

and $\delta \in [0, 1]$ is the depreciation rate. In our example, $y$ stands for $CO_2$ emissions, so that $\delta$ would stand for the natural reabsorption rate of $CO_2$ in the atmosphere.

## 3.1 Joint optimal choice before $T$

The first best solution for $t \in [0, +\infty)$ is obtained from the bloc's optimal control problem with objective function (7) and subject to the state equation (8).

**Via HJB**

**Proposition 1** *For any state trajectory $y(t)$, the optimal choices for player $i$ and $J$ are*

$$x_j^*(y) = a_j + B + Cy, \quad j = i, J,$$

*where*

$$C = \frac{r + 2\delta - \sqrt{(r+2\delta)^2 + 8b}}{2} (< 0), \quad B = \frac{(a_i + a_J)C}{r + \delta - 2C} (< 0).$$

*The optimal trajectory of state is:* $\forall t \geq 0$,

$$y(t) = (y_0 - y^*)e^{(2C-\delta)t} + y^*$$

*where $y^*$ is the optimal long-run steady state and given by*

$$y^* = \frac{a_i + a_J + 2B}{\delta - 2C}.$$

Thus the jointed social welfare, where the detail calculation is given in Appendix A.1:

$$\begin{aligned} W(\infty) &= \frac{(a_i^2 + a_J^2 - 2B^2)}{2r} - 2BC \int_0^{+\infty} e^{-rt} y(t) dt - (C^2 + b) \int_0^{+\infty} e^{-rt} y^2(t) dt \\ &= \frac{(a_i^2 + a_J^2 - 2B^2)/2 - 2BCy^* - (C^2+b)(y^*)^2}{r} + \frac{2BC(y_0 - y^*) + 2(C^2+b)y^*(y_0 - y^*))}{2C - \delta - r} \\ &\quad + \frac{(C^2+b)(y_0 - y^*)^2}{2(2C - \delta) - r}. \end{aligned} \tag{9}$$

If player i is not happy about the above optimal choice of the bloc and quits the bloc at time $T$, the above optimal choice continues until $t = T$ and state variable reaches

$$y(T) = (y_0 - y^*)e^{(2C-\delta)T} + y^* \tag{10}$$

with $T$ undetermined which depends on the choice of strategic space after the separation.

In the following, we illustrate two difference choices of strategic spaces after separation and demonstrate the importance of this different choices: The first case is where both players adopt Markovian strategy after separation and the second case is the situation where player J remains the initial commitments but player i adopts Markovian strategy when the separation happens.

The social welfare of player $i$ before the separation is thus

$$\begin{aligned}\alpha W(T) &= \alpha\left[\frac{(a_i^2+a_J^1-2B^2)(1-e^{-rT})}{2r} - 2BC\int_0^T e^{-rt}y(t)dt - (C^2+b)\int_0^T e^{-rt}y^2(t)dt\right] \\ &= \alpha\left[\frac{a_i^2+a_J^2-2B^2}{2} - 2BCy^* - (C^2+b)(y^*)^2\right]\frac{1-e^{-rT}}{r} \\ &\quad +\alpha\left[2BC(y_0-y^*)+2(C^2+b)y^*\,(y_0-y^*)\right]\;\frac{1-e^{(2C-\delta-r)T}}{2C-\delta-r} \\ &\quad +\alpha\,(C^2+b)\,(y_0-y^*)^2\;\frac{1-e^{(2(2C-\delta)-r)T}}{2(2C-\delta)-r}.\end{aligned} \tag{11}$$

Furthermore, it is straightforward that

$$\frac{dW(T)}{dT} = e^{-rT}\left[\frac{a_i^2+a_J^2-2B^2}{2} - 2BCy(T) - (C^2+b)y^2(T)\right] > 0 \tag{12}$$

if and only if

$$y(T) = (y_0-y^*)e^{(2C-\delta)T} + y^* \in (0,\ \ \underline{y})$$

where

$$\underline{y} = \frac{-2BC+\sqrt{4B^2C^2+2(C^2+b)(a_i^2+a_J^2-2B^2)}}{2(C^2+b)}(>0).$$

Remark. This is an interesting result, though I am still not clear the intuition behind.

Obviously, if initial condition checks

$$y_0 > \underline{y},$$

then $T=0$. Additionally, consider the situation where $y_0 < y^*$, then from the explicit solution, it is easy to see the accumulation of $y(t)$ is strictly increasing over time. Thus, if $y^* < \underline{y}$, separation will never happen, i.e., $T=+\infty$.

In the following, in order to study interior situation where separation happens in finite time, we impose parameter condition as following.

**Assumption 1** *Assume given parameters check the following inequality conditions:*

$$y_0 < \underline{y} < y^*.$$

## 3.2 Both player adopt Markovian strategy

Suppose player i exits the bloc at time $T^m$, the following existence of affine Markovian subgame perfect strategy can be obtained, whose proof is given by Appendix A.2.

**Proposition 2** *Suppose at time $T^m$ play $i$ quits from the bloc and after the separation both player $i$ and $J$ adopt Markovian strategy. Then there exists stable affine Markovian subgame perfect Nash equilibrium (MSPE)*

$$(x_i^m, x_J^m) = (a_i + B^m + C^m y,\ a_J + B^m + C^m y)\,, \ \ \forall y,$$

*with coefficients*

$$C^m = \frac{(r+2\delta) - \sqrt{(r+2\delta)^2 + 12b}}{6}(<0), \ \ B^m = \frac{(a_i + a_J)C^m}{r + \delta - 3C^m}(<0).$$

*For given initial condition at $T^m$, the corresponding optimal state trajectory is*

$$y^m(t) = (y(T^m) - \widehat{y^m})e^{(2C^m - \delta)(t - T^m)} + \widehat{y^m}, \ \ \forall t \geq T^m,$$

*where* $\widehat{y^m} = \dfrac{a_i + a_J + 2B^m}{\delta - 2C^m}(>0)$ *is asymptotically stable long-run steady state.*

It is easy to check that social welfare in the second period of player $i$ is

$$\begin{aligned} W^m_{i,II} &= \int_{T^m}^{+\infty} e^{-rt}\left(a_i x_i - \frac{x_i^2}{2} - \frac{by^2}{2}\right) dt \\ &= \frac{a_i^2-(B^m)^2}{2}\int_{T^m}^{+\infty} e^{-rt}dt - B^m\, C^m \int_{T^m}^{+\infty} e^{-rt} y^m(t)dt - \frac{((C^m)^2+b)}{2}\int_{T^m}^{+\infty} e^{-rt}(y^m)^2 dt \\ &= \left[\frac{a_i^2-(B^m)^2}{2} - \widehat{y^m} B^m C^m - \frac{(C^m)^2+b}{2}\widehat{y^m}^2\right]\frac{e^{-rT^m}}{r} \\ &\quad + \left[B^m C^m (y(T^m)-\widehat{y^m}) + ((C^m)^2+b)\widehat{y^m}\,(y(T^m)-\widehat{y^m})\right]\ \frac{e^{(2C^m-\delta-r)T^m}}{2C^m-\delta-r} \\ &\quad + \frac{((C^m)^2+b)}{2}\,(y(T^m)-\widehat{y^m})^2\ \frac{e^{(2(2C^m-\delta)-r)T^m}}{2(2C^m-\delta)-r}. \end{aligned} \tag{13}$$

It is straightforward to obtain the following results.[5]

**Corollary 1** *If production efficiency parameters $a_i$ and $a_J$ check $\left(2-\frac{r+\delta}{C^m}\right)a_i < a_J$, then separation will never happen: $T^m = +\infty$.*

Recall $a_i$ and $a_J$ measure production efficiency as mentioned above, this corollary states that if the bloc's advantage is sufficiently high, individual players have no incentive to quit the bloc. This is a rather straightforward result that the separation only can happen when some players feel no longer get sufficient benefits staying within the bloc.

In order to focus on the situation where separation happens in finite time, we impose the following assumption on the development parameters:

**Assumption 2** *Suppose*

$$\left(2-\frac{r+\delta}{C^m}\right)a_i > a_J.$$

Recall $C^m < 0$, the above sufficient condition of separation requires the leaving players' production efficiency should not be too low comparing to the remaining part. Obviously

[5]Proof. Similar to the previous subsection's arguments,

$$\frac{dW^m_{i,II}}{dT} = \frac{e^{-rT^m}}{2}[-a_i^2+(B^m)^2+2\,B^m\,C^m\,y(T^m)+((C^m)^2+b)y^2(T^m)] < 0$$

only if $-a_i^2+(B^m)^2<0$. Otherwise, we must have $\dfrac{dW^m_{i,II}}{dT}>0$. In other word, $T^m=+\infty$. Condition $-a_i^2+(B^m)^2>0$ is equivalent to $\left(2-\frac{r+\delta}{C^m}\right)a_i < a_J$.

if $a_i = a_J$, the above assumption is always true, thus separation happens in finite time for sure – no longer has advantage, thus incentive, to stay together.

Combing (11) and (13), the first order condition

$$\alpha \frac{dW(T)}{dT} + \frac{dW^m_{i,II}}{dT} = 0$$

yields the following results whose proof is given in the Appendix A.3.

**Proposition 3** *Let Assumption 1 and 2 hold. Suppose player i quits the bloc at time $T^m$ and after that player i and J adopt MSPE given by Proposition 2. Then if sharing parameter $\alpha$ checks*

$$\max\left\{\frac{(C^m)^2+b}{2(C^2+b)},\ \frac{a_i^2-(B^m)^2}{a_i^2+a_J^2-2B^2}\right\} \ <\ \alpha\ <\ 1, \tag{14}$$

*player i's unique optimal quitting time $T^m \in (0, +\infty)$ is given by*

$$T^m = \frac{1}{2C-\delta}\ln\left(\frac{y^m-y^*}{y_0-y^*}\right), \tag{15}$$

*where*

$$y^m = \frac{-\Sigma-\sqrt{\Sigma^2-4\Lambda\Gamma}}{2\Lambda}\,(\in (y_0,\ y^*))\,,$$

*with*

$$\Lambda = ((C^m)^2+b) - 2\alpha(C^2+b), \quad \Sigma = 2B^m\,C^m - 4\alpha BC$$

*and*

$$\Gamma = ((B^m)^2-a_i^2) + \alpha(a_i^2+a_J^2-2B^2).$$

If sharing condition (14) fails, either $T^m = 0$, separation starts immediately; or $T^m = +\infty$, no separation at all. The **surprising finding** is that it is not the small beneficial, rather the player, who has larger share in the aggregate welfare, would like to leave the bloc. Here, we do not model the relationship between contribution and share. If the share $\alpha$ is related to contribution, then condition (14) indicates that the main contributor of the bloc would like to quit the bloc in order to enjoy the full benefit from her efforts instead sharing with the others. If so, the condition (14) is no longer surprising.

## 3.3 Only player i recalculates her Markovian choices

Suppose player i exits the bloc at time $T^i$. Player $i$ face the following optimal control problem:

$$\max_{x_i} \int_{T^i}^{+\infty} e^{-rt} \left( a_i x_i - \frac{x_i^2}{2} - \frac{b\, y^2}{2} \right) dt,$$

subject to

$$\dot{y} = x_i + x_J^* - \delta y, \quad \forall t \geq T^i,$$

with $y(T^i) = (y_0 - y^*)e^{(2C-\delta)T^i} + y^*$ and $x_J^*(y) = a_J + B + Cy$ given.

**Via HJB**

We have the following results.

**Proposition 4** *Suppose at time $T^i$ player i quits the bloc and remaining player J stays with her initial commitment. Then the optimal Markovian strategy of player i is, for $t \geq T^i$,*

$$x_i^i(y) = a_i + B^i + C^i y, \quad \forall y. \tag{16}$$

*Furthermore, given initial condition $y(T)$, the corresponding state variable $y^i(t)$ is give by*

$$y(t)^i = (y(T^i) - \widehat{y^i})e^{(C^i+C-\delta)(t-T^i)} + \widehat{y^i} \;\; \forall t \geq T^i,$$

*where $\widehat{y^i}$ is the asymptotically stable long-run steady state and given by*

$$\widehat{y^i} = \frac{a_i + a_J + B + B^i}{\delta - C - C^i},$$

*and parameters*

$$C^i = \frac{r + 2\delta - 2C - \sqrt{(r + 2\delta - 2C)^2 + 4b}}{2} (< 0),$$

$$B^i = \frac{(a_i + a_J + B)C^i}{r + \delta - C - C^i} (< 0).$$

$$
\begin{aligned}
W^i_{i,II} &= \int_{T^i}^{+\infty} e^{-rt}\left(a_i x_i - \frac{x_i^2}{2} - \frac{by^2}{2}\right) dt \\
&= \frac{a_i^2 - (B^i)^2}{2}\int_{T^i}^{+\infty} e^{-rt}dt - B^i\, C^i \int_{T^i}^{+\infty} e^{-rt} y^i(t)dt - \frac{((C^i)^2+b)}{2}\int_{T^i}^{+\infty} e^{-rt}(y^i)^2 dt \\
&= \left[\frac{a_i^2-(B^i)^2}{2} - \widehat{y^i}B^iC^i - \frac{(C^i)^2+b}{2}\widehat{y^i}^2\right]\frac{e^{-rT^i}}{r} \\
&\quad + \left[B^iC^i(y(T^i)-\widehat{y^i}) + ((C^i)^2+b)\widehat{y^i}\,(y(T^i)-\widehat{y^i})\right]\frac{e^{(2C^i-\delta-r)T^i}}{2C^i-\delta-r} \\
&\quad + \frac{((C^i)^2+b)}{2}(y(T^i)-\widehat{y^i})^2\,\frac{e^{(2(2C^i-\delta)-r)T^i}}{2(2C^i-\delta)-r}.
\end{aligned} \tag{17}
$$

Similar to the above Corollary 1, if $\frac{dW^i_{i,II}}{dT} > 0$, player i will never quit the bloc. Thus , the following corollary is straightforward.[6]

**Corollary 2** *If production efficiency parameters $a_i$ and $a_J$ check $\left(\frac{2C(r+\delta-C)-CC^i}{C^i(r+\delta-C)} - \frac{r+\delta}{C^i}\right) a_i < a_J$, then separation will never happen: $T^i = +\infty$.*

It is easy to check that $C < C^m < C^i < 0$ and the production efficiency ratio checks [7] $\frac{a_J}{a_i} > \frac{2C(r+\delta-C)-CC^i}{C^i(r+\delta-C)} - \frac{r+\delta}{C^i} > 2 - \frac{r+\delta}{C^m} > 2$. In other words, **for given parameters setting and production efficiency, Corollary 1 and 2 indicate that it takes larger technology gap for player i to quit the bloc if she knows player J will be inactive after her quit. In other words, it is easier for player i to quit the bloc if both players adopt Markovian strategy than if player J is inactive.**

In the rest of this subsection, in order to study the situation where player i quits the bloc in finite time, we impose the following assumption

[6] The proof is the same as before. The necessary condition for separation happens, that is,

$$\frac{dW^i_{i,II}}{dT} = \frac{e^{-rT^m}}{2}[-\frac{a_i^2-(B^i)^2}{2} + B^i\,C^i\,y(T^i) + \frac{(C^i)^2+b}{2}y^2(T^i)] < 0,$$

is $-a_i^2 + (B^i)^2 < 0$. Thus, if the opposite is true, $(B^i)^2 - a_i^2 > 0$.

[7] The difference is

$$\left(\frac{2C(r+\delta-C)-CC^i}{C^i(r+\delta-C)} - \frac{r+\delta}{C^i}\right) - \left(2 - \frac{r+\delta}{C^m}\right) = 2\frac{(r+\delta-C)(C-C^i)-CC^i}{C^i(r+\delta-C)} + \frac{(r+\delta)(C^i-C^m)}{C^iC^m} > 0,$$

where both terms on the right hand side are positive due to $C < C^m < C^i < 0$ .

**Assumption 3** *Suppose*

$$\left(\frac{2C(r+\delta-C)-CC^i}{C^i(r+\delta-C)}-\frac{r+\delta}{C^i}\right)a_i > a_J.$$

This assumption is similar to the one in Assumption 2.

The first order condition of social welfare

$$\alpha\frac{dW(T)}{dT}+\frac{dW^i_{i,II}}{dT}=0$$

yields the following results whose proof is the same as the one for Proposition 3.

**Proposition 5** *Let Assumption 1 and 3 hold. Suppose player i quits the bloc at time $T^i$ and takes optimal choice as in Proposition 4 while player J adopts the same choice as in Proposition 1. Then if sharing parameter $\alpha$ checks*

$$\max\left\{\frac{a_i^2-(B^i)^2}{2(a_i^2+a_J^2-2B^2)},\ \frac{(C^i)^2+b}{4(C^2+b)}\right\}<\alpha<1, \tag{18}$$

*player i's unique optimal quitting time $T^i \in (0,+\infty)$ is given by*

$$T^i=\frac{1}{2C-\delta}\ln\left(\frac{y^i-y^*}{y_0-y^*}\right), \tag{19}$$

*where*

$$y^i=\frac{-\Sigma^i-\sqrt{(\Sigma^i)^2-4\Lambda^i\Gamma^i}}{2\Lambda^i}$$

*with*

$$\Lambda^i=\frac{(C^i)^2+b}{2}-2\alpha(C^2+b),\quad \Sigma^i=B^i\,C^i-4\alpha BC$$

*and*

$$\Gamma^i=\frac{(B^i)^2-a_i^2}{2}+\alpha(a_i^2+a_J^2-2B^2).$$

# 4 Comparison studies

## 4.1 A limited case

It is worth to study a limited special case where $a_i = a_J = 0$. In other words, we do not consider the gain from consumption, rather in the objective function, there are only costs: from emission, $x_j^2$ $(j = i, J)$, and from pollution accumulation, $y^2$.

Furthermore, given depreciation parameter $\delta$ is nonessential, we take $\delta = 0$. Then, it is straightforward,

$$x_j^* = Cy, \ \ \forall y, \ \ C = \frac{r - \sqrt{r^2 + 8b}}{2}, \ \ B = 0, \ \ y^* = 0,$$

and

$$y(t) = y_0 e^{2Ct},$$

which is strictly decreasing from $y_0$ to $y^* = 0$.

Obviously, the three Assumptions in the above subsections all fail and we can not directly apply the above findings.

Nonetheless, it is easy to get that

$$\alpha \frac{dW(T)}{dT} = -\alpha(C^2 + b) y_0^2 e^{(4C - r)t} < 0.$$

Thus, delay the separation deceases the social welfare of player $i$ (as well player $J$), thus the separation should happen immediate: $T = 0$.

However, if separation indeed happens at $T^m$ and after the separation, both players adopt Markovian strategy, then,

$$x_j^m = C^m y, \ \ \forall y, \ \ C^m = \frac{r - \sqrt{r^2 + 12b}}{6}, \ \ B^m = 0, \ \ \widehat{y^m} = 0,$$

and

$$y^m(t) = y(T^m) e^{2C^m (t - T^m)}.$$

It is straightforward,

$$\frac{dW^m(T)}{dT} = \frac{(C^m)^2 + b}{2} (y(T^m))^2 e^{-rT^m} > 0.$$

In other words, delay the separation will increase the afterward social welfare of both players, which yield $T^m = +\infty$, and that contradicts to the above $T = 0$.

From the first order condition[8]

$$\alpha \frac{dW(T)}{dT} + \frac{dW^m(T)}{dT} = \left[ -\alpha(C^2 + b) + \frac{(C^m)^2 + b}{2} \right] (y(T^m))^2 e^{-rT^m} = 0,$$

if and only if

$$\alpha = \alpha^m = \frac{(C^m)^2 + b}{2(C^2 + b)}.$$

Hence, the following results are proved.

**Proposition 6** *Let $a_i = a_J = 0$ and $\delta = 0$. If the two identical players decide to separate at $T^m$ from their coalition, and after the separation, both players adopt Markovian strategy, then,*

$$\begin{cases} T^m = 0, & \text{if} \ \ \alpha < \alpha^m, \\ T^m = +\infty, & \text{if} \ \ \alpha > \alpha^m. \end{cases}$$

In other words, the identical players' coalition either stays together forever or separate immediately at the beginning of the study.

A similar exercise can be done for the case where one player adopts Markovian strategy and the other one stays with the coalition's optimal choice, regardless the two players otherwise identical. Similar results of separation happens either at the beginning of the cooperation or never. More precisely,

$$\begin{cases} T^i = 0, & \text{if} \ \ \alpha < \alpha^i, \\ T^i = +\infty, & \text{if} \ \ \alpha > \alpha^i, \end{cases} ,$$

with $\alpha^i = \frac{((C^i)^2+b)/2}{2(C^2+b)}$ and $C^i = \frac{\sqrt{r^2+8b}-\sqrt{r^2+12b}}{2}$.

The above two different cases, defined as different strategic spaces after the separation, share similar separation patterns. The only difference relies on the threshold of share $\alpha$. It is easy to show that

$$\frac{(\alpha^i)^2 + b}{2} < (\alpha^m)^2 + b,$$

thus there is less chances for the coalition to stay together if both player adopt Markovian

[8]The second order condition is always zero and provides no more information of sufficiency.

strategy after separation. We can conclude the above study into the following:

$$\begin{cases} \text{if } \ 0 < \alpha < \alpha^i, \ \ T^m = T^i = 0, \\ \text{if } \ \alpha^i < \alpha < \alpha^m, \ \ T^m = 0, T^i = +\infty, \\ \text{if } \ \alpha^m < \alpha < 1, \ \ T^m = T^i = +\infty. \end{cases}$$

Thus, even this very special case of identical players, care should be given to the strategic choice after potential separation when study international agreement.

## 4.2 Marginal Bellman values –shadow values

It is easy to prove that $C < C^m < 0$ and $B < B^m < 0$. Thus

$$V'(y) = B + Cy < B^m + C^m y = V_j'(y), \ \ \forall y > 0, \ j = i, J.$$

In other words, as one union, the bloc is more sensitive to the increases of pollution than each individual player.

At the separation $T^m$,

$$V'(y(T^m)) < V_j'(y(T^m)), \ \ j = i, J.$$

## 4.3 Long-run steady states

The long-run steady states check the following relationship

**Corollary 3** *For given parameter $a_i, a_J, r, \delta$ and $b$, the 3 possible long-run steady states rank as*

$$\widehat{y^i} > \widehat{y^m} > y^*.$$

## 4.4 Switching times

From Proposition 3 and 5, it is straightforward that

$$T^m - T^i = \frac{1}{2C - \delta} \ln \left( \frac{y^m - y^*}{y^i - y^*} \right).$$

Given $C < 0$, thus $T^m > T^i$ if and only if $y^m < y^i$.

It is rather difficult to say, though $y^m$ and $y^i$ are explicit. Can we plot by numerical exercises to show that both cases: $T^m > T^i$ and $T^m < T^i$ are possible ?

### 4.5 Social welfare of player i, J and aggregate

## 5 Conclusion

The main contribution of this paper is that we demonstrated, via linear-quadratic differential game, the importance of strategic choices after the separation from a union or international organization. The choice of strategy matters not only for the moment of switching, the long-run steady state and essentially the individual and aggregate social welfare, but also for the choice before the separation. Thus it provides information when preparing a union or international treaty, cares should not only give to the period while the treaty is active, but also need to take into account the aftermath.

# A Appendix: Proof

## A.1 Proof of joint social welfare (9)

From the definition of joint social welfare, we have

$$
\begin{aligned}
W(\infty) = & \int_0^{+\infty} e^{-rt} \left[ a_i x_i + a_J x_J - \frac{x_i^2 + x_J^2}{2} - by^2 \right] dt \\
= & \int_0^{+\infty} e^{-rt} \left[ a_i(a_i + B + Cy) - \frac{(a_i + B + Cy)^2}{2} + a_J(a_J + B + Cy) - \frac{(a_J + B + Cy)^2}{2} - by^2 \right] dt \\
= & \int_0^{+\infty} e^{-rt} \left[ \frac{(a_i + B + Cy)}{2}(2a_i - (a_i + B + Cy)) + \frac{(a_J + B + Cy)}{2}(2a_J - (a_J + B + Cy)) - by^2 \right] dt \\
= & \int_0^{+\infty} e^{-rt} \left[ \frac{(a_i + B + Cy)(a_i - B - Cy)}{2} + \frac{(a_J + B + Cy)(a_J - B - Cy)}{2} - by^2 \right] dt \\
= & \int_0^{+\infty} e^{-rt} \frac{(a_i + B)(a_i - B) + (a_J + B)(a_J - B)}{2} dt \\
& + \int_0^{+\infty} e^{-rt} \left[ \frac{(a_i - B) - (a_i + B) + (a_J - B) - (a_J + B)}{2} Cy - C^2 y^2 - by^2 \right] dt \\
= & \frac{a_i^2 + a_J^2 - 2B^2}{2} \int_0^{+\infty} e^{-rt} dt - 2BC \int_0^{+\infty} e^{-rt} y dt - (C^2 + b) \int_0^{+\infty} e^{-rt} y^2 dt.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\int_0^{+\infty} e^{-rt} y dt = & \int_0^{+\infty} e^{-rt} [(y_0 - y^*) e^{(2C - \delta)t} + y^*] dt \\
= & \int_0^{+\infty} e^{-rt} (y_0 - y^*) e^{(2C - \delta)t} dt + \int_0^{+\infty} e^{-rt} y^* dt \\
= & (y_o - y^*) \int_0^{+\infty} e^{(2C - \delta - r)t} dt + y^* \int_0^{+\infty} e^{-rt} dt \\
& \frac{(y_o - y^*)(-1)}{2C - \delta - r} + \frac{y^*}{r}.
\end{aligned}
$$

Similarly,

$$
\int_0^{+\infty} e^{-rt} y^2 dt = \cdots = -\frac{(y_o - y^*)^2}{2(2C - \delta) - r} - \frac{2y^*(y_o - y^*)}{2C - \delta - r} + \frac{(y^*)^2}{r}.
$$

Substituting the above two explicit forms into $W(\infty)$ and rearranging terms, we obtain the expression of (9).

## A.2 Proof of Proposition 2

The proof is completed in three steps: step 1 demonstrates the existence of affine-linear Marvkovian Nash equilibrium; step 2 shows the stability and step 3 proves that the given affine-linear strategy is the unique.

**Remark for Ben. The uniqueness still need to proof, but similar to previous work.**

**Step 1. Existence of Markovian Nash equilibrium**

Define the Bellman Value function of player $j = i, J$ as $U_j(y)$, which must check the following HJB equation: for $t \geq T^m$,

$$rU_j(y) = \max_{x_j}\left[a_j x_j - \frac{x_j^2}{2} - \frac{b\,y^2}{2} + U_j'(y)\ \ (x_i + x_J - \delta y)\right],\ \ j = i, J.$$

Then the first order condition yields

$$x_j^m(t) = a_j + U_j'(y(t)). \tag{20}$$

Guess

$$U_j(y) = A_j + B_j y + \frac{C_j}{2}y^2, \quad \text{and} \quad j = i, J,$$

then

$$U_j'(y) = B_j + C_j y.$$

Substituting $x_i = a_i + B_i + C_i y$ and $x_J^m(t) = a_J + B_J + C_J y(t)$ into the HJB equations, comparing coefficients on both hand sides, it yields

$$\begin{cases} rA_i = \frac{(a_i+B_i)^2}{2} + (a_J + B_J)B_i, \\ (r + \delta - C_i - C_J)B_i = C_i(a_i + a_J + B_J), \\ (r + 2\delta)C_i = C_i^2 + 2C_i\,C_J - b, \end{cases} \tag{21}$$

and

$$\begin{cases} rA_J = \frac{(a_J+B_J)^2}{2} + (a_i + B_i)B_J, \\ (r + \delta - C_i - C_J)B_J = C_J(a_i + a_J + B_i), \\ (r + 2\delta)C_J = C_J^2 + 2C_i\,C_J - b, \end{cases} \tag{22}$$

**Remark**. More generally, if $b_i \neq b_J$, then the $b$ in the last two equations should be $b_i$ and $b_J$ respectively.

Solving the above two group equations system simultaneously, it follows that the only coefficients which yields valid Bellman value functions are

$$C_i = C_J = \frac{(r+2\delta) - \sqrt{(r+2\delta)^2 + 12b}}{6} \equiv C^m, \text{ and } B_i = B_J = \frac{(a_i + a_J)C^m}{r + \delta - 3C^m} \equiv B^m,$$

and

$$A_j^m = \frac{a_j^2}{2r} + \frac{(a_i + a_J)B^m}{r} + \frac{3(B^m)^2}{2r}, \quad j = i, J.$$

Thus the Markovian Nash equilibrium is given by

$$(x_i^m, x_J^m) = (a_i + U_i'(y),\ a_J U_J'(y)) = (a_i + B^m + C^m y,\ a_J + B^m + C^m y)\,, \quad \forall y.$$

**Step 2 Stability**

The stability is straightforward by substituting the above Markovian Nash equilibrium into the state equation, it yields

$$\dot{y} = (a_i + a_J + 2B^m) + (2C^m - \delta)y \quad \forall t \geq T^m$$

with $y(T^m)$ coming from the first period cooperation and $T^m$ unknown. The explicit solution is thus straightforward as given in the Proposition. Furthermore, it is easy to obtain that long-run steady state

$$y^m(t) = (y(T^m) - \widehat{y^m})e^{(2C^m - \delta)t} + \widehat{y^m}(> 0).$$

Given $2C^m - \delta < 0$, for any $y(T^m)$, the trajectory asymptotically converges to this steady state.

## A.3 Proof of Proposition 3

Suppose $-a_i^2 + (B^m)^2 < 0$ and Assumption 1 holds. The switching time $T^m$ must be given by the FOC

$$\alpha \frac{dW(T)}{dT} + \frac{dW_{i,II}^m}{dT} = 0,$$

provided the second order sufficient condition holds:

$$\alpha \frac{d^2 W(T^m)}{dT^2} + \frac{d^2 W^m_{i,II}(T^m)}{dT^2} < 0.$$

Substituting the explicit forms of $\frac{dW(T)}{dT}$ and $\frac{dW^m_{i,II}}{dT}$ into the first order condition, it yields

$$\alpha e^{-rT}\left[(a_i^2 + a_J^2 - 2B^2) - 4BCy(T) - 2(C^2 + b)y^2(T)\right]$$
$$+e^{-rT}\left[(B^m)^2 - a_i^2) + 2B^m\, C^m y(T) + ((C^m)^2 + b)y^2(T)\right] = 0.$$

Combining terms, the above equation can be rewritten as the following second degree polynomial in term of $y(T)$:

$$\Lambda\, y^2(T) + \Sigma\, y(T) + \Gamma = 0 \tag{23}$$

where

$$\Lambda = (C^m)^2 + b - 2\alpha(C^2 + b), \quad \Sigma = 2B^m\, C^m - 4\alpha BC$$

and

$$\Gamma = (B^m)^2 - a_i^2 + \alpha(a_i^2 + a_J^2 - 2B^2).$$

The roots, if exists, are

$$y^m(T) = \frac{-\Sigma \pm \sqrt{\Sigma^2 - 4\Lambda\Gamma}}{2\Lambda}. \tag{24}$$

Given parameters $\Lambda, \Sigma$ and $\Gamma$ are independent of switching time $T$, the second order sufficient condition holds if and only if

$$(2\Lambda y(T^m) + \Sigma)\;\; y'(T^m) < 0.$$

Given the assumption that pollution accumulation is increasing over time, that is, $y'(T^m) > 0$ is true always, then the second order sufficient condition holds if and only if

$$2\Lambda y(T^m) + \Sigma < 0. \tag{25}$$

Combing the second order condition (25) and the explicit solution (24), it follows that

$$2\Lambda y(T^m) + \Sigma = \pm\sqrt{\Sigma^2 - 4\Lambda\Gamma} < 0$$

if and only if the negative sign is taken in the explicit solution (24). Taking into account that only positive pollution level is possible, then

$$y^m(T) = -\frac{\Sigma + \sqrt{\Sigma^2 - 4\Lambda\Gamma}}{2\Lambda} > 0,$$

which is true either (i) $\Lambda < 0$, $\Gamma > 0$ (these two inequality implicitly guarantee the existence of real positive solution from FOC) and regardless the sign of $\Sigma$ or; (ii) $\Lambda > 0$, $\Gamma > 0$, $\Sigma < 0$ and provided $\Sigma^2 - 4\Lambda\Gamma \geq 0$ (the last condition guarantees the existence of real solution from FOC. If it fails, $T = +\infty$, separation never happens).

The case (i) corresponds to the situation where the first order condition yields unique positive root; while case (ii) corresponds to two positive roots from the FOC, in which one yields maximum welfare and the other one is related to minimum welfare. We study case by case.

**Case (i)**. Condition $\Lambda < 0$ is equivalent to

$$\alpha > \frac{(C^m)^2 + b}{2(C^2 + b)}$$

and $\Gamma > 0$ if and only if

$$\alpha > \frac{a_i^2 - (B^m)^2}{a_i^2 + a_J^2 - 2B^2}.$$

Combing the above two inequalities together, it follows that if and only if

$$\max\left\{\frac{(C^m)^2 + b}{2(C^2 + b)},\ \frac{a_i^2 - (B^m)^2}{a_i^2 + a_J^2 - 2B^2}\right\} < \alpha < 1, \tag{26}$$

the unique positive solution from the FOC (23), denoted as $y^m$, is

$$y^m = \frac{-\Sigma - \sqrt{\Sigma^2 - 4\Lambda\Gamma}}{2\Lambda}(> 0).$$

On the other hand,

$$y(T) = (y_0 - y^*)e^{(2C-\delta)T} + y^* = y^m.$$

Rearranging terms, it yields that

$$T^m = \frac{1}{2C - \delta} \ln\left(\frac{y^m - y^*}{y_0 - y^*}\right).$$

Recall Assumption 1 guarantees that $y_0 < y(T) < y^*$, thus, $0 < \frac{y^m - y^*}{y_0 - y^*} < 1$ and

$$T^m \in (0, +\infty).$$

**Case (ii)**. Similar to the case (i), condition $\Lambda > 0$ if and only of

$$\alpha < \frac{(C^m)^2 + b}{2(C^2 + b)},$$

$\Gamma > 0$ if and only if

$$\alpha > \frac{a_i^2 - (B^m)^2}{a_i^2 + a_J^2 - 2B^2}$$

and $\Sigma < 0$ if and only if

$$\frac{B^m C^m}{2BC} < \alpha.$$

Thus, the first condition is

$$\max\left\{\frac{B^m C^m}{2BC}, \ \frac{a_i^2 - (B^m)^2}{a_i^2 + a_J^2 - 2B^2}\right\} < \alpha < \frac{(C^m)^2 + b}{2(C^2 + b)}. \tag{27}$$

Now consider the last condition $\Sigma^2 - 4\Lambda\Gamma \geq 0$, which guarantees the existence of solution from the FOC. If it fails, then for any $\alpha$, separation will never happen, $T^m = +\infty$.

$$\Sigma^2 - 4\Lambda\Gamma \geq 0$$

if and only if

$$\alpha \in (0, \underline{\alpha}] \bigcup [\overline{\alpha}, 1), \tag{28}$$

where $\underline{\alpha}$ and $\overline{\alpha}$ are the two roots of second degree of polynomial

$$\overline{A}\alpha^2 + \overline{B}\alpha + \overline{C} = 0,$$

with

$$\overline{A} = 4B^2C^2 + 2(a_i^2 + a_J^2 - 2B^2)(C^2 + b)(> 0), \ \ \overline{C} = (B^m)^2(C^m)^2 - ((C^m)^2 + b)((B^m)^2 - a_i^2)(> 0)$$

and

$$\overline{B} = -4BC(B^m)^2(C^m)^2 + 2((C^2+b)((B^m)^2 - a_i^2) - (a_i^2 + a_J^2 - 2B^2)((C^m)^2 + b) < 0.$$

Combining the condition (27) and (28) together, it follows that if and only if

$$\max\left\{\frac{B^mC^m}{2BC},\ \frac{a_i^2-(B^m)^2}{a_i^2+a_J^2-2B^2},\ \overline{\alpha}\right\} < \alpha < \frac{(C^m)^2+b}{2(C^2+b)}, \tag{29}$$

the FOC has positive solution $y^m$ which yields maximum social welfare for player i.

The above condition (29) seemingly extends the exiting interval of $\alpha$ obtained in case (i). Nonetheless, this extension depends on the relationship between $\frac{a_i^2-(B^m)^2}{a_i^2+a_J^2-2B^2}$, $\frac{(C^m)^2+b}{2(C^2+b)}$ and the combination on the left-hand-side of (29).

If $\frac{a_i^2-(B^m)^2}{a_i^2+a_J^2-2B^2} > \frac{(C^m)^2+b}{2(C^2+b)}$, then condition (29) is an empty interval, that is, case (ii) can not happen. And condition (26) is reduced to

$$\frac{a_i^2-(B^m)^2}{a_i^2+a_J^2-2B^2} < \alpha < 1.$$

If $\frac{a_i^2-(B^m)^2}{a_i^2+a_J^2-2B^2} < \frac{(C^m)^2+b}{2(C^2+b)}$, then (29) rather shrinks the definition domain from case (i), given its left-hand-side condition. Combing condition (29) and (26) yields

$$\max\left\{\frac{a_i^2-(B^m)^2}{a_i^2+a_J^2-2B^2},\ \frac{a_i^2-(B^m)^2}{a_i^2+a_J^2-2B^2}\right\} < \alpha < 1.$$

which is the sufficient condition. That completes the proof.

## A.4 Proof of Proposition 4

Define the Bellman Value function as $V_i(y)$, which must check the following Hamilton-Jacob-Bellman (HJB) equation: for $t \geq T^i$,

$$rV_i(y) = \max_{x_i}\left[a_ix_i - \frac{x_i^2}{2} - \frac{b\,y^2}{2} + V_i'(y)\ \ (x_i + x_J^* - \delta y)\right],$$

where $x_J^*$ is given in Proposition 1,

$$x_J^* = a_J + B + Cy.$$

Given the linear-quadratic forms of the objective function and linear state equation, we can guess that the Bellman value function follow affine-quadratic form as:

$$V_i(y) = A_i + B_i y + \frac{C_i^2}{2}\, y^2,$$

Taking first order condition on the right hand side of the HJB equation, it yields the optimal choices

$$x_i(y) = a_i + V_i'(y) = a_i + B_i + C_i y, \quad \forall t \geq T^i.$$

Substituting the optimal choice into the right hand side of the HJB equation, we have

$$\begin{aligned}
RHS &= a_i(a_i + B_i + C_i y) - \frac{(a_i + B_i + C_i y)^2}{2} - \frac{by^2}{2} \\
&\quad + (B_i + C_i y)\left[(a_i + B_i + C_i y) + (a_J + B + Cy) - \delta y\right] \\
&= a_i(a_i + B_i) + a_i C_i y - \frac{1}{2}\left[(a_i + B_i)^2 + 2C_i(a_i + B_i)y + C_i^2 y^2\right] - \frac{by^2}{2} \\
&\quad + (B_i + C_i y)[(a_i + a_J + B_i + B) + (C_i + C - \delta)y] \\
&= a_i(a_i + B_i) - \frac{(a_i + B_i)^2}{2} + B_i(a_i + a_J + B_i + B) \\
&\quad + \left[a_i C_i - C_i(a_i + B_i) + C_i(a_i + a_J + B_i + B) + B_i(C_i + C - \delta)\right] y \\
&\quad + \left[-\frac{C_i^2 + b}{2} + C_i(C_i + C - \delta)\right] y^2 \\
&= \frac{a_i^2 - B_i^2}{2} + B_i^2 + B_i(a_i + a_J + B) + \left[C_i(a_i + a_J + B) + B_i(C_i + C - \delta)\right] y \\
&\quad + \left[-\frac{C_i^2 + b}{2} + C_i(C_i + C - \delta)\right] y^2.
\end{aligned}$$

Comparing coefficients on both sides of the HJB equation, it follows

$$\begin{cases} rA_i = \frac{(a_i+B_i)^2}{2} + (a_J + B)\ B_i, \\ (r+\delta - C - C_i)B_i = (a_i + a_J + B)\ C_i, \\ (r+2\delta)C_i = C_i^2 + 2C\ C_i - b. \end{cases} \tag{30}$$

The last equation yields one positive and one negative roots. Given the valid Bellman value function must be concave, so we take only the negative one:

$$C_i = \frac{r + 2\delta - 2C - \sqrt{(r+2\delta-2C)^2 + 4b}}{2}.$$

Given $C < 0$ and $b > 0$, it is straightforward, $(r+2\delta-2C)^2 + 4b > (r+2\delta-2C)^2 > 0$. Thus

$$C_i < 0.$$

Furthermore,

$$B_i = \frac{(a_i + a_J + B)C_i}{r + \delta - C - C_i}(< 0).$$

Substituting the above optimal choice of player i and $x_J^*$ into the state equation, it follows

$$\dot{y} = (a_i + a_J + B + B_i) + (C_i + C - \delta)y \quad \forall t \geq T^i$$

with $y(T^i)$ from Proposition 1. The solution is straightforward.

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