Resolving Child Custody Disputes Efficiently

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Abstract

We study arbitration mechanisms for child custody arrangements in the shadow of a default court ruling that will be made without any consideration of the parents' preferences. We show that the ideal default custody, providing the best chances for the efficiency of arbitration, is not an extreme custody regime which assigns full custody to one of the parents. Instead it is the custody arrangement that maximizes the reservation payoffs of the most difficult parent types to persuade to forfeit this default custody arrangement and accept arbitration. As long as all involved parties' utility functions are concave, this result also generalizes to the multiple-parties cases where custody settlements require the consent of more parties than the two parents. Further, when parents have the quadratic disutility function, efficient arbitration is possible only under the ideal default custody, regardless of the parents' type distributions. This possibility result extends to the case where custody decisions involve more issues/dimensions than the share of the time that the children will spend with each parent. It also extends to the multiple-parties setting. Finally, we identify a class of utility functions that do not allow efficient arbitration even under the ideal default custody.

Introduction

In this paper we study arbitration mechanisms for child custody arrangements. A custody arrangement most importantly determines the proportion of the time that the child/children will spend with each of the parents after a divorce/separation. The parents' preferences are not necessarily monotonic in the time that they spend with the child. In our model, these preferences are represented with concave utility functions, where the most preferred custody arrangement of each parent is her private information. Parents transmit messages indicating their preferences to the arbitration mechanism, which in turn sets the custody regime as well as the monetary transfers between the parents in the form of alimony, child support, and allocation of common property. The alternative to arbitration is a default custody ruling that will be made by a court according to a legal standard, without any consideration of the divorcing parents' preferences. By participating in the arbitration mechanism, the parents forfeit this default custody arrangement.

An *ex-post efficient* custody regime is defined as the arrangement that maximizes the sum of the parents' utilities from custody. The efficiency label is justified with the assumption that the parents' payoffs are linear in monetary transfers – although not linear in child custody arrangements – and the common belief that the parents take the best interests of the child into account when forming preferences. Here, we examine the efficient arbitration mechanism that implements this efficient custody arrangement subject to the incentive compatibility, individual rationality, and budget balance constraints.

There is a rich literature on efficient mechanism design with voluntary participation which goes back to the studies of efficiency in bilateral trade and partnership dissolution. As established by this literature, implementing the efficient custody arrangement instead of using the default regime would generate a value added for each of the two parents. However, this implementation would also require that the parents share their private information with the arbitration mechanism, which would necessitate some information rent to be left for the parents on top of what they would get under the outside option. It is one of the main findings of the efficient design literature that there exists an efficient mechanism if and only if the value added generated by efficiency is large enough to cover the information rent. The difference between the value added and the information rent can be interpreted as the maximized revenue of a mechanism designer constrained to offer efficient mechanisms (Krishna and Perry, 1998 and Williams, 1999).

In the context of divorce settlements, the magnitudes of the value added from efficiency and the information rent are both influenced by the default custody regime determined by the legal standard. In this paper we study the existence of default custody arrangements permitting efficient arbitration mechanisms. Answering this existence question involves finding the *ideal default custody* arrangement that maximizes the revenue of the constrained mechanism designer introduced above.

In our child custody setting, we show that both the ideal default custody arrangement and the *critical parent types* associated with this default (which are defined as the most difficult types to persuade to opt out of this default custody arrangement and accept arbitration instead) are bounded away from the boundary points in the policy and type sets. In other words, an extreme custody regime which assigns full custody to one of the parents and deprives the other parent from all the rights on the child will not be an ideal default arrangement to support efficient arbitration. This actually confirms what we have learned from the bilateral trade/partnership dissolution settings: There is no efficient bilateral trade mechanism when the alternative to trade is the seller keeping the entire good to herself (Myerson and Satterthwaite, 1983), whereas partnerships can be dissolved efficiently when the alternative to dissolution is an equitable division of the assets (Cramton, Gibbons, and Klemperer, 1987).

What is surprising in our setting is that the ideal default custody, which provides the best chances for the efficiency of arbitration, is the custody arrangement that *maximizes* the sum of the *reservation payoffs* of the critical types of the two parents at the default custody arrangement. Even though the revenue of the constrained mechanism designer is convex in default custody for fixed critical types, once the endogeneity of the critical types is taken into consideration, the revenue turns out to be a function that attains a maximum for an interior custody level. It follows from the envelope theorem that the latter function is maximized only if the former one is minimized at an interior solution. In other words, the ideal default custody is the efficient custody from the perspective of the parent types that are most obstinate to give up this default and accept arbitration. If an outside observer does not recognize the endogeneity of these obstinate types in the chosen default custody regime, ironically she might come to the misleading conclusion that the ideal default custody is in fact chosen to minimize the potential for arbitration.

In keeping with the earlier literature on efficient mechanism design, we focus on arbitration mechanisms that aim to implement the efficient custody arrangement from the two divorcing parents' perspective. This is also in line with the assessment of many law scholars who consider the two parents as much better advocates for the interests of their child than any judge or policymaker (see for example Mnookin and Kornhauser, 1979). Our analysis, however, can be extended to implementation of any custody arrangement that is monotonic and continuous in the types of the parents. The ideal default custody, which would facilitate implementation of such arrangements, should also maximize the reservation payoffs of the critical parent types. Some real-life custody settlements may require the consent of more parties than the two divorcing parents, such as the grandparents, grown-up siblings, and attorneys or the state representing the children. As long as all involved parties' utility functions are concave, our result that the ideal default custody maximizes the reservation payoffs of the critical types generalizes to this multiple-parties setting as well.

The observation that the ideal default custody is the *efficient* custody from the perspective of the critical parent types allows us to write down the maximized revenue from the constrained mechanism design problem as a function of these critical types. The properties of this function lead to novel possibility and impossibility results on the existence of efficient mechanisms.

A particular specification of our model is supported by the *quadratic disutility* function, where each parent's payoff is quadratically decreasing in the *distance* between the chosen custody and her most preferred custody arrangement. In this setting, we show that efficient arbitration is possible only under the ideal default custody. The ideal custody here is identical to the ex-ante efficient custody that the court would have chosen in the absence of arbitration.[1] This result is a *general possibility result* in the sense that it holds for any continuous parent type distribution with full support. For the policymaker, it is sufficient to know the expected types of the two parents in order to identify the ideal default custody that uniquely permits efficient arbitration.

Custody decisions may also involve more dimensions than the share of the time that the children will spend with each parent (e.g., multiple children, how the crucial decisions for the child's future such as school choice/extracurricular activities will be made, etc.). Our possibility result for quadratic disutility functions extends to this multi-dimensional environment: An efficient arbitration mechanism exists only under the ideal default custody, which is also the ex-ante efficient custody choice of the court. This possibility result holds in the multiple-parties setting as well.

We complement the possibility result with a similarly general *impossibility result* that we derive in an alternative setting. Suppose each parent's (dis)utility from custody is

[1] Ex-ante efficient default rules are known to enhance ex-post efficient bargaining when the negotiating parties have non-concave utility functions as well. See Che (2006) and Segal and Whinston (2016) for linear utility functions, Segal and Whinston (2011) for convex utility functions.

determined by the magnitude of the *difference* between the chosen custody and the parent's most preferred arrangement. Quadratic disutility model is a special case of this specification because *distance* is the absolute value of difference. When the second derivative of the parents' utility function is convex and strictly monotonic (either increasing or decreasing), we show that there is no default custody that permits efficient bargaining, regardless of the parameters of the continuous type distributions.

Our questions and setup have a lot in common with the recent state that the legal profession has reached regarding the process of divorce and custody bargaining. Today, a vast majority of divorcing couples resolve their custody disputes outside of the court, either with the help of a mediator/arbitrator or by more informal means of bargaining. The seminal judicial work by Mnookin and Kornhauser (1979) asked "how the rules/procedures used in court for adjudicating disputes would affect the bargaining process that occurs between divorcing couples *outside* the courtroom." Although they did not dwell on the forms or specifics of divorce/custody bargaining, Mnookin and Kornhauser had very clear ideas as to what custodial bargaining would involve: Two main elements would be money and custody, which would be inextricably linked in that "over some range of alternatives, each parent may be willing to exchange custodial rights and obligations for income or wealth." The preferences with regard to custody can vary a lot among individuals, if not among the genders in general. More specifically, many parents may have single-peaked preferences regarding time spent on child-rearing responsibilities, but their peaks may occur at different points. The arguments of Mnookin and Kornhauser dominated the legal profession and their work helped paving the way for the shift from adversarial to non-adversarial resolution of divorce-related parenting disputes.

From a strict Coasian perspective, the laws and procedures – which regulate the outside options of the negotiating parties – should not have an effect on the efficiency of the final settlement. As argued by Becker (1981) in his *Treatise on the Family*, the impact of the changing legal standards should be detected only on the division of the generated value between the parties. Peters (1986) observed that the introduction of the no-fault grounds for termination of marriages in the US did not change the American divorce rates and considered this as an evidence for the Coase theorem. Other researchers questioned the applicability of the Coase theorem to divorces by pointing to the high transaction costs between the partners to a marriage to be dissolved. They provided evidence to the impact of the laws and procedures on not only the divorce rates (Allen, 1992 and Friedberg, 1998)[2]

[2] See also Wolfers (2006) and Gonzalez and Viitanen (2009) for the continuation of the debate for the

but also on the probability of reaching an out-of-court settlement (Brinig and Alexeev, 1993) and the nature of the custody arrangements in these settlements (Allen and Brinig, 2011). For instance, Brinig and Alexeev observed that the out-of-court settlement rates are higher for divorcing couples in Wisconsin, which applied a joint-custody standard, than in Virginia, where courts were more likely to give custody to the primary caretakers under the child's best interest standard. Brinig (2006) attributed this observation to the possibility that parents consider the joint-custody standard as a *penalty default* with little resemblance to what they really want and concluded that this standard is adopted with the aim of forcing the parents to settle outside of the court.

One important source of transaction costs in divorce settlements is that the parents' preferences regarding child custody are generally private information to them, especially after facing new and different monetary and psychological circumstances and perspectives in the post-divorce era. It is not possible for an arbitrator to know ex ante if the father prefers joint custody to maternal custody. As in any bargaining/negotiation environment, the parties would have a tendency to misrepresent preferences to approach further to their ideal outcome following a divorce: In interviews with Californian divorcees, Maccoby and Mnookin (1992) find that 10% of mothers and 7% of fathers requested for more physical custody in their initial divorce petitions than they actually wanted. Our study contributes to the debate on the assessment of the practical adjudication procedures for child custody disputes by integrating this asymmetric information aspect into the parent's pre-trial bargaining. Our results suggest that the default rules adopted by the courts have an impact on the efficiency of the pre-trial negotiation not only through the determination of the parents' reservation payoffs in case of a negotiation failure, but also through their effect on the information rent that these parents would get at the successful completion of these negotiations. Once the informational asymmetry between the bargaining parents is taken into account, we find that negotiations are more likely to result in an efficient settlement under default rules that the parents (at least the critical types of the parents) would find desirable, rather than under penalty-like defaults.

impact of the divorce laws on the rates of divorce. Stevenson and Wolfers (2006) provide evidence on the impact of the changes in divorce laws on domestic violence, suicide, and spousal murder rates.

1 The Model

We refer to the two parents as parent 1 and parent 2.[3] The nature of the custody is represented by $x \in [0,1]$, with the interpretation that x is the proportion of time that the child (or children) will spend with parent 1 after the divorce/separation; hence $1-x$ is the proportion of time that will be spent with parent 2. The preferred custody arrangement from each parent's perspective is her private information. This private information is represented by parent i's type $\theta_i \in [0,1]$. In addition to the custody arrangement, the two parents care about the monetary transfers $t_1, t_2 \in R$ that they receive or make. The payoff function for parent i is

$$u_i(x, t_i, \theta_i) = v_i(x, \theta_i) + t_i,$$

where v_i is a twice continuously differentiable direct utility function of parent i with type θ_i from custody arrangement x. Function v_i is strictly concave in x (i.e., $\frac{\partial^2 v_i(x,\theta_i)}{\partial x^2} < 0$) and its cross partial derivative is positive (i.e., $\frac{\partial^2 v_i(x,\theta_i)}{\partial x \partial \theta_i} > 0$). It is maximized in x when $x = \theta_i$. The types of the parents are independently distributed on $[0,1]$. The distribution functions are continuous and they have full support.

As an example to these preferences, consider the situation where each parent's direct utility from custody is determined by the difference between the implemented and the desired custody arrangements: $v_i(x - \theta_i)$, where v_i is a concave function maximized at 0. A special case for this would be the quadratic disutility function such that $v_i(x, \theta_i) = -(x - \theta_i)^2$. We will come back to these preference specifications to prove our possibility and impossibility results.

Following the notation in Segal and Whinston (2011, 2012, 2016) papers, we define the surplus generated with the custody decision x as the sum of the direct utility functions of the parents:

$$s(x, \theta_1, \theta_2) = v_1(x, \theta_1) + v_2(x, \theta_2).$$

Strict concavity of functions v_1 and v_2 imply that function s is strictly concave in x as well, hence there is a unique custody level $x^*(\theta_1, \theta_2)$ that maximizes this surplus. We refer to this custody arrangement as the **ex-post efficient custody.** We also define the maximized surplus as a function of the types of the two parents:

$$S(\theta_1, \theta_2) = \max_x s(x, \theta_1, \theta_2) = s(x^*(\theta_1, \theta_2), \theta_1, \theta_2).$$

[3] Our model applies to custody agreements between both opposite-sex and same-sex parents.

In this paper, we are interested in arbitration mechanisms that implement the ex-post efficient custody. Invoking the revelation principle, we model arbitration as a direct revelation mechanism: The parents reveal their types θ_1 and θ_2 to the mechanism and the arbitrator sets the corresponding ex-post efficient custody level $x^*(\theta_1, \theta_2)$ together with transfers $t_1(\theta_1, \theta_2)$ and $t_2(\theta_1, \theta_2)$. Each parent has the option to refuse to participate in this mechanism and opt for the court-determined default ruling. Applying the legal standard, the court imposes a default custody $x_0 \in [0, 1]$ which does not depend on the types of the parents. We follow the normalization that the transfer payment that will be decreed by the court is zero. In other words, t_i can be interpreted as the difference between the transfer from the arbitration mechanism and the transfer from a potential court decision.

We say that the default custody arrangement x_0 **permits efficient arbitration** if there exist transfer functions $t_1(\theta_1, \theta_2)$ and $t_2(\theta_1, \theta_2)$ which satisfy the following individual rationality, incentive compatibility, and budget balance conditions together with the ex-post efficient custody arrangement $x^*(\theta_1, \theta_2)$. An efficient arbitration mechanism is

- **individually rational** if each parent prefers arbitration to the default custody x_0:

$$IR : E_{\theta_j}\left[v_i\left(x^*\left(\theta_i, \theta_j\right), \theta_i\right) + t_i\left(\theta_i, \theta_j\right)\right] \geq v_i\left(x_0, \theta_i\right) \text{ for all } \theta_i \in [0, 1] \text{ and } i = 1, 2;$$

- **incentive compatible** if each parent prefers to reveal her type truthfully:

$$IC : E_{\theta_j}\left[v_i\left(x^*\left(\theta_i, \theta_j\right), \theta_i\right) + t_i\left(\theta_i, \theta_j\right)\right] \geq E_{\theta_j}\left[v_i\left(x^*\left(\theta_i', \theta_j\right), \theta_i\right) + t_i\left(\theta_i', \theta_j\right)\right]$$
$$\text{for all } \theta_i, \theta_i' \in [0, 1] \text{ and } i = 1, 2;$$

- **budget balanced** if monetary transfers add up to zero for all type pairs:

$$BB : t_1(\theta_1, \theta_2) + t_2(\theta_1, \theta_2) = 0 \text{ for all } \theta_1, \theta_2 \in [0, 1];$$

  where operator E_{θ_j} refers to the expectation over the type of parent j.

The earlier literature asks analogous questions on the existence of efficient mechanisms in bilateral trade and partnership dissolution settings, where $v_i(x, \theta_i)$ is linear both in the allocation decision x and the agent type θ_i. Myerson and Satterthwaite (1983) establish that there is no efficient bilateral trade mechanism that secures allocating the good to the buyer whenever her valuation for it is higher than that of the seller. We learn from the work of Cramton, Gibbons, and Klemperer (1987) that this impossibility result relies on the extreme nature of the default option: The seller keeps the entire good to herself in case of a trade failure. An efficient allocation mechanism would exist as long as the default alternative to accepting the mechanism is a more equitable division. For instance, there is an efficient mechanism allocating the sole ownership of a firm to the partner who has the highest valuation for it, provided that no partner has a very large initial share in the firm.[4] These earlier results point to the importance of the default custody x_0 for the arbitrator's ability to mediate an efficient custody arrangement for the parents. Krishna and Perry (1998) and Williams (1999) extend the *efficient mechanism design* analysis to more general preference functions and type sets.[5] We now briefly sketch the analysis of this earlier literature, applying it to our concave-utility setting. We report the main result from this analysis in Lemma 1.

As a first step to the assessment of the existence of an ex-post efficient mechanism, we drop the budget balance requirement and consider the revenue maximization of a mechanism designer who maximizes $-t_1 - t_2$ subject to the individual rationality and incentive compatibility constraints, as well as the requirement that the resulting custody is efficient $x^*(\theta_1, \theta_2)$. Because different parent types assess the default custody differently, this is an example to a maximization problem with type-dependent reservation payoffs as in Jullien (2000). Thanks to the efficiency requirement and our differentiability assumptions on the parent's utility functions, any incentive compatible mechanism is an *expected externality* mechanism: The expected transfer to parent i with type θ_i is identified by the expected direct utility of the other parent $E_{\theta_j} v_j(x^*(\theta_i, \theta_j), \theta_j)$ up to a constant term.[6] A transfer equaling exactly to this term would give a parent with type $\hat{\theta}_i$ an expected payoff equal to the maximized surplus $E_{\theta_j} S\left(\hat{\theta}_i, \theta_j\right)$. Considering that this parent would forego the reservation payoff of $v_i\left(x_0, \hat{\theta}_i\right)$ by accepting the arbitration mechanism, she would be indifferent to agreeing to arbitration if her expected transfer is set to

$$E_{\theta_j} v_j(x^*(\theta_i, \theta_j), \theta_j) - E_{\theta_j} S\left(\hat{\theta}_i, \theta_j\right) + v_i\left(x_0, \hat{\theta}_i\right).$$

The expectation of this term over θ_i gives us the expected transfer $E[t_i]$ to parent i when the default custody arrangement is x_0 and type $\hat{\theta}_i$ of parent i is indifferent between accepting the mechanism and the default custody:

$$E[t_i] = E_{\theta_i} E_{\theta_j} v_j(x^*(\theta_i, \theta_j), \theta_j) - E_{\theta_j} S\left(\hat{\theta}_i, \theta_j\right) + v_i\left(x_0, \hat{\theta}_i\right).$$

[4] For efficient mechanism design with linear utility, also see Che (2006), Ornelas and Turner (2007), Figueroa and Skreta (2012), Yenmez (2012), Agastya and Birulin (2018), Loertscher and Wasser (2019).

[5] Also see Makowski and Mezetti (1994), Neeman (1999), Schweizer (2006), Segal and Whinston (2011, 2012, 2016).

[6] See Arrow (1979) and d'Aspremont and Gérard-Varet (1979) for expected externality mechanisms.

Negative of the sum of these transfers for the two parents yields the expected revenue of a mechanism designer offering this efficient mechanism:

$$\pi\left(x_0,\hat{\theta}_1,\hat{\theta}_2\right)=E_{\theta_2}S\left(\hat{\theta}_1,\theta_2\right)+E_{\theta_1}S\left(\theta_1,\hat{\theta}_2\right)-E_{\theta_1\theta_2}S\left(\theta_1,\theta_2\right)-s\left(x_0,\hat{\theta}_1,\hat{\theta}_2\right).$$

For the individual rationality condition to be satisfied by this mechanism, all types of both parents must (weakly) prefer to accept the mechanism. Accordingly, the maximized expected revenue of the mechanism designer from an incentive compatible and individually rational mechanism is

$$\bar{\pi}\left(x_0\right)=\min_{\hat{\theta}_i,\hat{\theta}_2}\pi\left(x_0,\hat{\theta}_1,\hat{\theta}_2\right).$$

Because the types of the parents are drawn from the closed and bounded set $[0,1]$, the above minimization problem is well defined. The types $\hat{\theta}_1\left(x_0\right)$, $\hat{\theta}_2\left(x_0\right)$ solving this problem are called the **critical types** which are the most difficult ones to persuade to participate in this arbitration mechanism. For future reference, we note the first-order necessary condition for the critical type $\hat{\theta}_i$ under default decision x_0 below:

$$E_{\theta_j}\frac{\partial v_i\left(x^*\left(\hat{\theta}_i,\theta_j\right),\hat{\theta}_i\right)}{\partial\theta_i}-\frac{\partial v_i\left(x_0,\hat{\theta}_i\right)}{\partial\theta_i}\begin{array}{l}\geq 0 \text{ for } \hat{\theta}_i=0\\ =0 \text{ for } \hat{\theta}_i\in(0,1)\\ \leq 0 \text{ for } \hat{\theta}_i=1\end{array}\tag{1}$$

It follows from the previous literature on efficient mechanism design that there exists a budget-balanced efficient arbitration mechanism under x_0 if and only if the maximized revenue of this constrained mechanism is non-negative.

Lemma 1 *Default custody arrangement x_0 permits efficient arbitration if and only if $\bar{\pi}\left(x_0\right)$ is non-negative.*

The constrained mechanism constructed above is already incentive compatible and individually rational. As long as this constrained mechanism does not run an expected deficit, the degrees of freedom associated with Bayesian implementation can be used to balance the budget, so that the transfers sum up to zero under all possible type pairs. Here is one way to interpret this result: Allocating the custody efficiently – instead of letting the court impose a default custody – generates some value added for the parents. But an efficient arbitration mechanism should leave some information rent to these parents, so that they are induced to share their private information with the mechanism, ensuring that the decided custody arrangement is indeed efficient. If the constrained revenue maximizing mechanism is not running a deficit, it means that the added value is high enough to cover the required information rent.

In the context of linear direct utility functions, Myerson and Satterthwaite (1983) establish that extreme default arrangements are not compatible with efficient mechanism design. With our first proposition, we confirm that their insight extends to our setting with concave direct utility functions. If the default custody takes an extreme value, then there is no efficient arbitration mechanism.

Proposition 1 *Extreme values of the default custody ($x_0 = 0$ and $x_0 = 1$) do not permit efficient arbitration.*

Proof. Suppose $x_0 = 0$. To see that the critical types are also zero under this extreme default custody, consider the first derivative of $\pi\left(0, \hat{\theta}_1, \hat{\theta}_2\right)$ defining the critical type $\hat{\theta}_i(0)$ in (1). It follows from the positive cross partial derivative of function v_i that the left hand side of (1) is positive for all $\hat{\theta}_i$. This observation establishes that $\hat{\theta}_i(0) = 0$ for $i = 1, 2$. When both parents have type 0, the ex-post efficient custody maximizing the sum of their utility functions is also zero, implying that $S(0,0) = s(0,0,0)$. Hence we can write $\bar{\pi}(0)$ as the following expectation:

$$E_{\theta_1 \theta_2}\left[S(0, \theta_2) + S(\theta_1, 0) - S(\theta_1, \theta_2) - S(0,0)\right].$$

The expression in the square brackets takes value zero at $\theta_1 = \theta_2 = 0$. To see that this expression is strictly decreasing in both θ_1 and θ_2, consider its first derivative with respect to θ_i. Because x^* is chosen optimally, it follows from the envelope theorem that this derivative is equal to

$$\frac{\partial v_i\left(x^*(\theta_i, 0), \theta_i\right)}{\partial \theta_i} - \frac{\partial v_i\left(x^*(\theta_i, \theta_j), \theta_i\right)}{\partial \theta_i}.$$

Recall that x^* is a strictly increasing function. It follows from the positive cross partial derivative condition on v_i that the derivative in the above display is negative for any value of θ_j other than 0. This implies that the expression in the square brackets above is negative for almost every θ_1, θ_2 pair. Therefore its expectation is also negative and $\bar{\pi}(0) < 0$.

A similar proof can be constructed to show that $\hat{\theta}_i(1) = 1$ and $\bar{\pi}(1) < 0$. ■

The proposition above does not rule out efficient mechanisms that can be supported by intermediate default custody arrangements. The same way that the default of an equitable division can sustain an efficient allocation of the assets of a dissolving firm, an intermediate default custody could facilitate the parents' agreement on an efficient custody arrangement. We would like to explore the existence of such default custody arrangements.

More specifically, our aim is establishing efficient arbitration results which do not depend on the specifics of the type distributions of the two parents. At this juncture, we remark that earlier possibility results on efficient mechanisms do not apply to our custody allocation setting with concave utility functions. Cramton, Gibbons, and Klemperer (1987) focus exclusively on linear utility. And the generalizations by Schweizer (2006) and Segal and Whinston (2011) cover cases with payoffs that are convex in the decision variable x or with default arrangements that can be randomized.[7] Segal and Whinston's (2016) impossibility result does not apply to our setting either, since the parents do not have *efficient opt-out types* whose refusal of arbitration will result in an efficient custody allocation.

We investigate the existence of default custody levels permitting efficient arbitration by examining the sign of function $\bar{\pi}(x_0)$ at its maximum level. This is a similar exercise to the analyses of Che (2006), Schweizer (2006), Figueroa and Skreta (2012), Segal and Whinston (2016), and Loertscher and Wasser (2019).[8] We call the default custody that maximizes $\bar{\pi}(x_0)$ and therefore gives the best chances for an efficient arbitration the **ideal default custody** $\hat{x}_0$. The following result reveals an intriguing property of this ideal default.

Proposition 2 *The ideal default custody* $\hat{x}_0$ *is in the interior of* $[0,1]$ *and it maximizes surplus* $s\left(x, \hat{\theta}_1(\hat{x}_0), \hat{\theta}_2(\hat{x}_0)\right)$ *for the critical parent types:*

$$\hat{x}_0 = x^*\left(\hat{\theta}_1(\hat{x}_0), \hat{\theta}_2(\hat{x}_0)\right).$$

Proof. We first show that the ideal default custody cannot be a boundary point of $[0,1]$. As in the proof of the earlier proposition, we start with considering the first derivative of $\pi\left(x_0, \hat{\theta}_1, \hat{\theta}_2\right)$ defining the critical type $\hat{\theta}_i(x_0)$ in (1). If x_0 is close enough to 0, it follows from the positive cross partial derivative of function v_i that the left hand side of (1) is positive for all $\hat{\theta}_i$. That is, for small enough values of x_0, critical types $\hat{\theta}_1(x_0)$ and $\hat{\theta}_2(x_0)$ are constant at 0. For these levels of default custody, $\bar{\pi}(x_0)$ equals a constant minus $s(x_0, 0, 0)$. Because $s(x_0, 0, 0)$ is decreasing in x_0, function $\bar{\pi}(x_0)$ is increasing at $x_0 = 0$. Therefore $x_0 = 0$ cannot be the ideal default custody which is defined as the arrangement maximizing $\bar{\pi}(x_0)$. A similar argument shows that $\bar{\pi}(x_0)$ is decreasing at $x_0 = 1$ and hence $x_0 = 1$ cannot be the ideal default custody either.

[7] We will address the issue of randomized default custody in our Concluding Remarks section.

[8] A similar maximization problem appears in the design of the optimal auction in the presence of externalities among the bidders. The auction designer chooses how to threaten the bidders in case that they refuse to participate. See Jehiel, Moldovanu, and Stacchetti (1999) for multi-dimensional types and Figueroa and Skreta (2009, 2011) for single-dimensional private information.

It follows from the envelope theorem that the first derivative of $\bar{\pi}(x_0)$ is equal to the first derivative of $\pi\left(x_0, \hat{\theta}_1, \hat{\theta}_2\right)$ with respect to x_0 where $\hat{\theta}_1$ and $\hat{\theta}_2$ are the critical types under x_0, because the former function is the lower envelope of the latter one:

$$\bar{\pi}'(x_0) = \pi_{x_0}\left(x_0, \hat{\theta}_1(x_0), \hat{\theta}_2(x_0)\right) = -s_{x_0}\left(x_0, \hat{\theta}_1(x_0), \hat{\theta}_2(x_0)\right).$$

For an interior extreme point of function $\bar{\pi}(x_0)$, this derivative is zero. Because s is a concave function of x_0, it attains a maximum for such a value of x_0. ■

The proposition does not provide a closed-form solution for the ideal default custody, because $\hat{x}_0$ appears in both sides of the equality in its statement. Instead, it yields a necessary condition for the ideal default custody, which has interesting implications and which will be important in the derivation of the rest of our results.

We can see that the default custody x_0 affects the constrained revenue of the mechanism designer $\pi\left(x_0, \hat{\theta}_1, \hat{\theta}_2\right)$ in two ways: First, x_0 enters directly in the reservation utility of the parents with the critical types $s\left(x_0, \hat{\theta}_1, \hat{\theta}_2\right)$. This reservation utility has a negative sign in the constrained revenue function. Second, a variation in x_0 also changes the critical types $\hat{\theta}_1(x_0)$ and $\hat{\theta}_2(x_0)$, which also enter into function π. Because $\bar{\pi}(x_0)$ is defined as the lower envelope of the $\pi\left(x_0, \hat{\theta}_1, \hat{\theta}_2\right)$ functions, it follows from the envelope theorem that the second effect does not have an impact on the first derivative of $\bar{\pi}(x_0)$. Accordingly, a local extreme point for $\bar{\pi}(x_0)$ is also a local extreme point for $\pi\left(x_0, \hat{\theta}_1, \hat{\theta}_2\right)$, hence for $s\left(x_0, \hat{\theta}_1, \hat{\theta}_2\right)$, provided that $\hat{\theta}_i = \hat{\theta}_i(x_0)$.

What is surprising about the proposition is that maximization of $\bar{\pi}(x_0)$ implies minimization of $\pi\left(x_0, \hat{\theta}_1, \hat{\theta}_2\right)$ for $\hat{\theta}_i = \hat{\theta}_i(x_0)$. It follows from the concavity of the utility functions v_i that each $\pi\left(x_0, \hat{\theta}_1, \hat{\theta}_2\right)$ is a *convex* function of x_0. Yet, the lower envelope of these functions, $\bar{\pi}(x_0) = \pi\left(x_0, \hat{\theta}_1(x_0), \hat{\theta}_2(x_0)\right)$, is not convex. It has a maximum in the interior of the custody policy space $[0, 1]$. By maximizing $\bar{\pi}(x_0)$, the ideal default custody minimizes $\pi\left(x_0, \hat{\theta}_1, \hat{\theta}_2\right)$. (See Figure 1.)

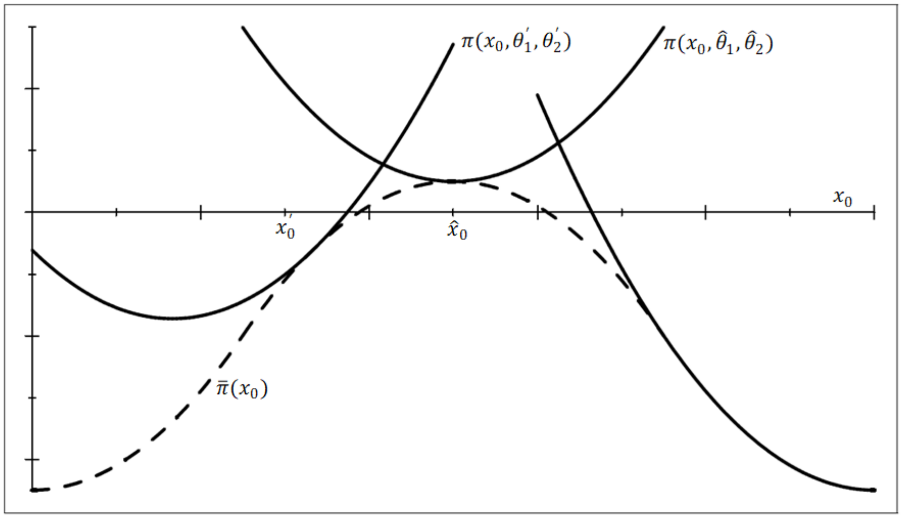

Figure 1: Function $\bar{\pi}(x_0)$ as the lower envelope of $\pi(x_0, \cdot)$.

This is an unexpected observation. The default custody $\hat{x}_0$ constitutes the outside option of participation in the arbitration mechanism for the two parents. Yet, maximization of the designer's revenue – and therefore creating the value to be distributed to the parents during a budget-balanced arbitration – involves maximizing the magnitude of this outside option, for the types of the parents which are most difficult to persuade to participate in the mechanism. As alluded to in the Introduction, after identifying these critical types, a naive observer might think that the default custody $\hat{x}_0$ is set to incite the parents' refusal of the arbitration process. This reasoning of course misses the endogeneity of the critical types on the default custody.

This observation also sheds some light on Segal and Whinston's (2012) assessment that reducing surplus from the default option for the critical types makes efficient negotiation more likely. Their assessment holds for all default custody arrangements other than the ideal one, because $d\bar{\pi}(x_0) = \frac{\partial\pi\left(x_0, \hat{\theta}_1(x_0), \hat{\theta}_2(x_0)\right)}{\partial x_0}$. However, under the ideal default custody, $d\bar{\pi}(x_0)$ equals 0 and the impact of a change in the default custody is determined by the second order effects, which suggest different directions for $\bar{\pi}(x_0)$ and $\pi\left(x_0, \hat{\theta}_1, \hat{\theta}_2\right)$.

Proposition 2 has an interesting implication for divorce arbitration. Starting with Mnookin and Kornhauser (1979), many legal scholars consider the joint-custody legal standard as a default arrangement that both parents would like to avoid, "very much like ...

Solomon's threat to cut the child in half." When comparing divorce settlements in two different states, Brinig and Alexeev (1992) observe that parents are more likely to settle out of court in Wisconsin – which followed the joint-custody standard – than in Virginia where the family courts had a tendency to give full custody to the primary caretaker before separation. Brinig (2006) interprets this finding as a confirmation that parents are unlikely to have a preference for joint custody and policymakers may use it as a penalty default[9] to incite out-of-court settlements. Our result suggests a completely different consideration for setting legal standards. Once the informational asymmetries are taken into account, we show that encouraging parents to go through an efficient arbitration process requires a legal standard which is closer to what is desired by the parents, especially by the parent types which are more difficult to persuade to take part in arbitration.

To explain the idea further, imagine that the ideal default custody rule that gives the best chances for an efficient out-of-court settlement is indeed a joint-custody arrangement ($\hat{x}_0$ in Figure 1). According to Proposition 2, joint custody is also the most efficient custody arrangement from the perspective of the critical types of the parents who are the most reluctant to give up the court-ordered custody and accept an ex-post efficient arbitration mechanism. Now suppose that a naive policymaker shifts the default custody rule towards a paternal custody standard (such as x_0' in Figure 1), with the hope that the parents would find this custody arrangement less desirable and hence they would be easier to persuade to take part in arbitration. Under this alternative default rule, the (original) critical parent types would indeed have lower reservation payoffs. However, their compliance with the efficient mechanism would require that they do not pretend to have a higher preference for paternal custody. This would imply an obligation to leave a higher information rent to these parents during negotiations. Accordingly, the policy shift to a less desirable default rule could put efficient arbitration in jeopardy, because the surplus generated by the ex-post efficient allocation of the custody may not be sufficient to cover these higher information rents.

Remark 1 (*Linear direct utility*) Because of the strict concavity assumption, our setting does not nest the partnership dissolution models where the partners' payoffs are linear in the shares they hold in the partnership (Cramton, Gibbons, and Klemperer, 1987). Nevertheless, we can see that the above proposition is valid for these models as well. Our default custody rule x_0 is analogous to the initial shares of the partners in the partnership setting. When critical types are held constant, function π is linear in these initial shares,

[9] See Ayres and Gertner (1989) for a general discussion of penalty defaults.

with a slope equal to the derivative of the lower envelope function $\bar{\pi}$. Under the "ideal" initial shares that maximize function $\bar{\pi}$, function π is constant and attains its extreme value in a trivial sense.

When function π is constant, it means that the total payoff cannot be increased by shifting the firm's shares from one partner with the critical type to another partner with the critical type. This implies that under the ideal allocation of the initial shares, critical types of all the partners should be the same. This is indeed the condition identified by Che (2006) and Figueroa and Skreta (2012) for the initial allocation of shares to give the highest chances for an efficient dissolution.

Remark 2 (*Alternative custody functions*) Our focus in this paper is the mechanisms implementing the ex-post efficient custody allocation $x^*(\theta_1, \theta_2)$ from the perspective of the two parents. One justification for this is the assessment of many law scholars who consider the two parents as much better advocates for the interests of their child than any judge or policymaker (Mnookin and Kornhauser, 1979). Nevertheless, it is worth noting that Proposition 2 holds for the implementation of alternative custody functions $x'(\theta_1, \theta_2)$ that are monotonic and continuous in both dimensions.[10] The ideal default custody $\hat{x}_0'$ that would maximize the potential rent for a mechanism implementing $x'(\theta_1, \theta_2)$ satisfies

$$\hat{x}_0' = x^*\left(\hat{\theta}_1\left(\hat{x}_0'\right), \hat{\theta}_2\left(\hat{x}_0'\right)\right),$$

where $\hat{\theta}_1(\hat{x}_0')$ and $\hat{\theta}_2(\hat{x}_0')$ are the critical types that are most difficult to persuade to accept the mechanism instead of the default $\hat{x}_0'$. Monotonicity of $x'(\theta_1, \theta_2)$ is needed for the incentive compatibility condition to be satisfied and its continuity ensures that the envelope theorem applies. Custody function $x'(\theta_1, \theta_2)$ here can be considered either as an exogenous arrangement maximizing the child's welfare in response to the preferences of the two parents, or as an endogenous one that solves a maximization problem. Examples to such maximization problems are the arbitrator's revenue maximization, maximization of efficiency subject to a budget constraint (Segal and Whinston, 2016), or a combination of revenue and efficiency (Loertscher and Wasser, 2019).[11]

[10] Following the arguments in Segal and Whinston (2011), a constrained revenue function $\pi\left(x_0, \hat{\theta}_1, \hat{\theta}_2\right)$ can be constructed for any such custody function. $\pi\left(x_0, \hat{\theta}_1, \hat{\theta}_2\right)$ has the sum of the reservation utilities of the two parents $s\left(x_0, \hat{\theta}_1, \hat{\theta}_2\right)$ as a negative term. As in Proposition 2, maximizing $\bar{\pi}(x_0) = \min_{\hat{\theta}_1, \hat{\theta}_2} \pi\left(x_0, \hat{\theta}_1, \hat{\theta}_2\right)$ implies maximizing $s\left(x_0, \hat{\theta}_1, \hat{\theta}_2\right)$ as well.

[11] Loertscher and Wasser (2019) make a similar point in a model with linear payoffs and interdependent

Remark 3 (*More than 2 parties to custody arrangements*) Some custody settlements require the agreement of additional parties such as the grandparents, grown-up siblings, teachers, social workers, and attorneys for the children. All these parties may have substantial claims on custody matters. The previous result generalizes to I (> 2) disputants, as long as their utility functions are concave in the custody arrangement $x \in [0,1]$ which is defined as the proportion of time that the child spends with parent 1. Let θ be the profile of the types of all I agents. We can define

$$\begin{aligned} \pi\left(x_0, \hat{\theta}\right) &= \sum_i E_{\theta_{-i}} S\left(\hat{\theta}_i, \theta_{-i}\right) - (I-1) E_\theta S(\theta) - s\left(x_0, \hat{\theta}\right), \\ \bar{\pi}(x_0) &= \min_{\hat{\theta}} \pi\left(x_0, \hat{\theta}\right). \end{aligned}$$

We know from the earlier literature on efficient mechanism design that an analogous result to Lemma 1 holds: Default arrangement x_0 permits efficient arbitration if and only if $\bar{\pi}(x_0)$ is non-negative. It follows from our analysis that x_0 *maximizing* $\bar{\pi}(x_0)$ also *minimizes* $\pi\left(x_0, \hat{\theta}\right)$, and hence maximizes $s\left(x_0, \hat{\theta}\right)$, when $\hat{\theta} = \hat{\theta}(x_0)$.

Remark 4 (*Multiple dimensions of custody)* Imagine that the two parents have to make decisions on n different dimensions of custody (decisions involving multiple children, or multiple dimensions of custody parameters for the same child such as school choices, extracurricular activities, etc.). One way to model this is to assume that the custody decision x is a vector which is an element of set $[0,1]^n$. In this extension of our model, parent types θ_1 and θ_2 are elements of $[0,1]^n$ as well, indicating their personal preferences on these dimensions. The payoff function of parent i is

$$u_i(x, t_i, \theta_i) = v_i(x, \theta_i) + t_i,$$

where v_i represents the direct utility of parent i with type θ_i from custody level x. We maintain the assumptions that v_i is concave in vector x (that is, $v_{i_{xx}} < 0$), its cross partial derivative is positive $\left(v_{i_{x\theta_i}} > 0\right)$, and it is maximized in x when $x = \theta_i$. The ex-post efficient custody arrangement $x^*(\theta_1, \theta_2)$ is now defined as a vector function.

We know from the earlier literature that Lemma 1 applies to this multidimensional extension of our model as well. That is, an efficient arbitration mechanism exists under default custody $x_0 \in [0,1]^n$ if and only if $\bar{\pi}(x_0)$ is non-negative. However, we cannot rule

values. They show that, even though the optimal allocation rule maximizing a convex combination of revenue and efficiency is endogenous, the expected values of the critical types of the partners are "equalized" under the ideal default allocation.

out the possibility that the ideal default policy maximizing $\bar{\pi}(x_0)$ is a corner solution. Therefore we cannot prove an analogous result to Proposition 2. The proposition would hold if the ideal default policy is in the interior of $[0,1]^n$. We will have more remarks on this point in the context of the quadratic disutility functions.

2 Constrained Revenue as Function of Critical Types

We start this section by showing that, the same way that the ideal default custody is in the interior of the policy space, the critical types of the two parents under this ideal custody are also in the interior of the type space.

Proposition 3 *Under the ideal default custody $\hat{x}_0$, the critical types $\hat{\theta}_1(\hat{x}_0)$ and $\hat{\theta}_2(\hat{x}_0)$ are in the interior of* $[0,1]$.

Proof. Suppose $\hat{\theta}_i(\hat{x}_0) = 0$. Consider again the first derivative of $\pi\left(\hat{x}_0, \hat{\theta}_i, \hat{\theta}_j\right)$ defining $\hat{\theta}_j(\hat{x}_0)$:

$$E_{\theta_i} \frac{\partial v_j\left(x^*\left(\theta_i, \hat{\theta}_j\right), \hat{\theta}_j\right)}{\partial \theta_j} - \frac{\partial v_j\left(x^*\left(0, \hat{\theta}_j\right), \hat{\theta}_j\right)}{\partial \theta_j}.$$

Because x^* is an increasing function of θ_i, it follows from the cross partial derivative condition on v_j that the difference term above is positive for all $\hat{\theta}_j$. Accordingly $\hat{\theta}_j(\hat{x}_0) = 0$ as well. The second part of Proposition 2 implies that $\hat{x}_0 = x^*\left(\hat{\theta}_i(\hat{x}_0), \hat{\theta}_j(\hat{x}_0)\right) = 0$. This, however, is a contradiction to the first part of the proposition which states $\hat{x}_0$ is in the interior of $[0,1]$.

A similar contradiction arises for $\hat{\theta}_i(\hat{x}_0) = 1$ as well. ∎

We learn from the analysis in the previous section that, when function $\bar{\pi}(x_0)$ is maximized, its last term reduces to the maximized surplus for the critical parent types under the ideal default custody:

$$s\left(\hat{x}_0, \hat{\theta}_1(\hat{x}_0), \hat{\theta}_2(\hat{x}_0)\right) = S\left(\hat{\theta}_1(\hat{x}_0), \hat{\theta}_2(\hat{x}_0)\right).$$

This allows us to rewrite the maximized constrained revenue $\max_{x_0} \bar{\pi}(x_0)$ as a function of these critical types:

$$\bar{\pi}(\hat{x}_0) = E_{\theta_2} S\left(\hat{\theta}_1, \theta_2\right) + E_{\theta_1} S\left(\theta_1, \hat{\theta}_2\right) - E_{\theta_1 \theta_2} S(\theta_1, \theta_2) - S\left(\hat{\theta}_1, \hat{\theta}_2\right),$$

where $\hat{\theta}_i = \hat{\theta}_i(\hat{x}_0)$. With some abuse of notation, we refer to the right hand side of the equation above as $\bar{\pi}\left(\hat{\theta}_1, \hat{\theta}_2\right)$. Because $x^*\left(\hat{\theta}_1, \hat{\theta}_2\right)$ maximizes function $s\left(x_0, \hat{\theta}_1, \hat{\theta}_2\right)$, function $\bar{\pi}\left(\hat{\theta}_1, \hat{\theta}_2\right)$ defines a lower envelope of $\pi\left(x_0, \hat{\theta}_1, \hat{\theta}_2\right)$ over the $\hat{\theta}_1 \times \hat{\theta}_2$ domain:

$$\bar{\pi}\left(\hat{\theta}_1, \hat{\theta}_2\right) = \min_{x_0} \pi\left(x_0, \hat{\theta}_1, \hat{\theta}_2\right) = \pi\left(x^*\left(\hat{\theta}_1, \hat{\theta}_2\right), \hat{\theta}_1, \hat{\theta}_2\right).$$

This function will be useful in the derivation of further results about efficient arbitration. It follows from the envelope theorem that any interior local extrema of function $\bar{\pi}(x_0)$ also constitutes an interior *saddle point* for $\bar{\pi}\left(\hat{\theta}_1, \hat{\theta}_2\right)$ satisfying the first-order conditions for interior local extrema, where $\hat{\theta}_i = \hat{\theta}_i(x_0)$.[12]

In other words, the first-order necessary conditions for interior local extrema of $\bar{\pi}\left(\hat{\theta}_1, \hat{\theta}_2\right)$ coincide with the first-order necessary conditions for the minimization problem defining the critical types under the ideal default custody $\hat{x}_0$. By examining the value that function $\bar{\pi}\left(\hat{\theta}_1, \hat{\theta}_2\right)$ takes in these saddle points, we can see whether the ideal default custody permits efficient arbitration. This observation will be critical in the derivation of our possibility and impossibility results for particular specifications of our model in the following sections.

3 A Possibility Result under Quadratic Disutility

As a notable special case of our model, consider the utility function $v_i(x, \theta_i) = -(x - \theta_i)^2$. With this broadly-used utility function, each parent's disutility from deviating from her personally preferred custody level is quadratically increasing in the magnitude of the deviation. Under this quadratic disutility specification, the surplus generated by custody x_0 is $s(x_0, \theta_1, \theta_2) = -(x_0 - \theta_1)^2 - (x_0 - \theta_2)^2$ and its maximized level with the ex-post efficient custody $x^*(\theta_1, \theta_2) = \frac{\theta_1 + \theta_2}{2}$ is

$$S(\theta_1, \theta_2) = \max_x s(x, \theta_1, \theta_2) = -2\left(\frac{\theta_2 - \theta_1}{2}\right)^2.$$

Proposition 4 *Under the quadratic disutility specification, the unique default custody that permits efficient arbitration is the average of the expected types of the two parents* $\frac{E\theta_1 + E\theta_2}{2}$.

Proof. We start with searching for an interior saddle point for function $\bar{\pi}\left(\hat{\theta}_1, \hat{\theta}_2\right)$. The derivative of this function with respect to $\hat{\theta}_i$ is

$$\begin{aligned} & E_{\theta_j} \frac{\partial v_i\left(x^*\left(\hat{\theta}_i, \theta_j\right), \hat{\theta}_i\right)}{\partial \theta_i} - \frac{\partial v_i\left(x^*\left(\hat{\theta}_i, \hat{\theta}_j\right), \hat{\theta}_i\right)}{\partial \theta_i} \\ = \; & 2E_{\theta_j} x^*\left(\hat{\theta}_i, \theta_j\right) - 2x^*\left(\hat{\theta}_i, \hat{\theta}_j\right) \\ = \; & E\theta_j - \hat{\theta}_j. \end{aligned}$$

[12] $\bar{\pi}'(x_0) = \pi_{x_0}\left(x_0, \hat{\theta}_1(x_0), \hat{\theta}_2(x_0)\right) = 0 \Rightarrow \pi_{\hat{\theta}_i}\left(x^*\left(\hat{\theta}_1, \hat{\theta}_2\right), \hat{\theta}_1, \hat{\theta}_2\right) = 0 \Rightarrow \bar{\pi}_{\hat{\theta}_i}\left(\hat{\theta}_1, \hat{\theta}_2\right) = 0.$

Therefore the unique interior saddle point of this function, where the first order conditions are satisfied as equalities, is $\hat{\theta}_i = E\theta_i$ for $i = 1, 2$. Proposition 2 implies that the ideal default custody is $\hat{x}_0 = \frac{E\theta_1 + E\theta_2}{2}$. Plugging $\hat{\theta}_i = E\theta_i$ in $\bar{\pi}\left(\hat{\theta}_1, \hat{\theta}_2\right)$ yields zero as the maximized level of the constrained revenue. Accordingly, there exists an efficient custody mechanism when the default custody is $\frac{E\theta_1 + E\theta_2}{2}$. But the constrained revenue is negative for any other default custody, ruling out the possibility of efficient arbitration for these alternative default custody arrangements. ■

We prove this result, by showing that the critical type of each parent under the ideal default custody is her expected type, i.e., $\hat{\theta}_i(\hat{x}_0) = E\theta_i$. It then follows from Proposition 2 that the ideal default custody is the average of these critical types. Function $\bar{\pi}\left(\hat{\theta}_1, \hat{\theta}_2\right)$ assumes value zero when the critical types equal the expected types. Notice that the result holds regardless of the specifics of the distribution functions of the parent types, as long as these distributions satisfy the full support and continuity conditions that we assumed from the outset. Moreover, to identify the ideal default custody, under which the efficient arbitration is possible, it is sufficient to know the expectations of these distribution functions. Nevertheless, the possibility result here does not go as far as the result by Cramton, Gibbons, and Klemperer (1987) which identifies a positive measure of initial shares that lead to efficient dissolution of partnerships.

The ideal default custody, which uniquely permits efficient arbitration under quadratic disutility, maximizes the surplus for the critical types of the two parents. Because the critical types are the average types, this also corresponds to maximizing the ex-ante expected surplus under the prior type distributions. In other words, to ameliorate the conditions for efficient arbitration, we do not need a strategic court system with the power to commit to a suboptimal custody level in case that arbitration fails. A benevolent judge would choose the unique custody level that would make efficient arbitration possible, as long as she does not update her beliefs about the parent types in case of an off-path refusal to participate in arbitration.

The efficient custody mechanism takes the form of an expected externality mechanism à la Arrow (1979) and d'Aspremont and Gérard-Varet (1979). Therefore it is easy to calculate the corresponding transfers that would support the efficient arrangement under the ideal default custody:

$$t_i(\theta_i, \theta_j) = \left(\frac{\theta_j - E\theta_i}{2}\right)^2 - \left(\frac{\theta_i - E\theta_j}{2}\right)^2 + h_i(\theta_i, \theta_j),$$

where $h_1(\theta_1, \theta_2)$ is an arbitrary function such that its expectation is zero over either one of its arguments and $h_2 = -h_1$. Budget balance of the mechanism follows from the symmetry of the transfers for the two agents. Incentive compatibility follows from the second term in the transfer. Individual rationality is satisfied as an equality for the critical type $E\theta_i$, and therefore it is strictly satisfied for all the other types. Unlike the alimony/child-support payments used in practice in many jurisdictions, the transfers associated with the efficient arbitration mechanism are not pinned down by the custody arrangement reached through arbitration alone. The preferences of the two divorcing parents that lead to this custody arrangement also enter into the determination of which parent makes the payment and its magnitude. In order to motivate the parents reveal their preferences to the mechanism, the parent who is more conciliatory with respect to the expected position of the ex-partner (in other words, the parent whose type is closer to the expected type of the other parent) receives a positive transfer (in expectation) from the less conciliatory one. Same as for the identification of the ideal default custody $\hat{x}_0$, the expected parent types – rather than the entire distribution functions – are sufficient for constructing these transfers (by setting function h_i constant at zero).

Remark 5 (*Ex-ante efficiency*) Ex-ante efficient default rules are known to enhance ex-post efficient bargaining when the negotiating parties have non-strictly-concave utility functions as well. When the two bargaining parties have linear utility functions and the policymaker is constrained to choose complete ownership by one of the parties as the default rule, Segal and Whinston (2016) show that a similar ex-ante efficient ownership rule is the ideal default rule that maximizes the difference between the value added from ex-post efficiency and the information rent. They argue that a similar result holds when the default rule is restricted to be a liability rule where one of the parties have the option to buy at a fixed price. See also Che (2006) on the latter result. In either case however, even the ideal default rule does not permit an efficient bargaining mechanism.

When the bargaining parties have convex utility functions, Segal and Whinston (2011) show that the ex-ante efficient default rule permits efficient bargaining, even if it is not the ideal default rule maximizing the difference between the value added of efficiency and the information rent. This result applies in the absence of transferable payoffs as well.

Remark 6 (*More than 2 parties to custody arrangements*) Same as the result in Proposition 2, the possibility result on quadratic disutility carries on in a model with more than 2 disputing parties. As long as all the I parties have a quadratic disutility for deviating from their most preferred custody levels, the unique default custody that admits efficient arbitration is the average of the expected types of all the disputants ($\sum_i E\theta_i \diagup I$) and the critical type of each disputant is her expected type.

Remark 7 (*Multiple dimensions of custody*) If each parent's disutility is quadratic in the Euclidean distance between the chosen custody x and her preferred custody θ_i, Proposition 4 continues to hold in a multi-issue environment, where custody decision x and the parent types θ_i are elements of the n-dimensional set $[0,1]^n$. Suppose that each parent i's direct utility from custody is

$$v_i(x,\theta_i) = -(dist(x,\theta_i))^2 = -(x_1-\theta_{i1})^2 - (x_2-\theta_{i2})^2 - ... - (x_N-\theta_{in})^2.$$

Because the square of the distance is separable in different dimensions of the policy space, the arbitration process can deal with each dimension of custody separately and Proposition 4 holds for each of these dimensions, proving the result for the multi-dimensional extension.

4 An Impossibility Result

It follows from our earlier analysis that the sign of function $\bar{\pi}\left(\hat{\theta}_1,\hat{\theta}_2\right)$ calculated at the critical types under the ideal default custody $\hat{x}_0$ determines whether there exists an efficient arbitration mechanism. The critical types $\hat{\theta}_1(\hat{x}_0)$ and $\hat{\theta}_2(\hat{x}_0)$ under the ideal default custody constitute an interior saddle point for this function, satisfying the first-order conditions for an extremum. By rearranging terms, we rewrite $\bar{\pi}\left(\hat{\theta}_1,\hat{\theta}_2\right)$ as

$$\left[E_{\theta_2}S\left(\hat{\theta}_1,\theta_2\right) - S\left(\hat{\theta}_1,\hat{\theta}_2\right)\right] - E_{\theta_1}\left[E_{\theta_2}S(\theta_1,\theta_2) - S\left(\theta_1,\hat{\theta}_2\right)\right].$$

At an interior saddle point $\left(\hat{\theta}_1,\hat{\theta}_2\right)$ of this function, $\hat{\theta}_1$ satisfies the first-order necessary condition for minimization of the terms in the first square brackets above, given $\hat{\theta}_2$. The impossibility result that we develop in this section will rely on establishing that $\hat{\theta}_1$ is the unique solution to this minimization problem. Hence, when evaluated at this saddle point, the expression in the first square brackets would be strictly smaller than the one in the second square brackets for *almost all* values of θ_1 (when $\theta_1 = \hat{\theta}_1$, the two expressions are identical). This would imply that $\bar{\pi}\left(\hat{\theta}_1,\hat{\theta}_2\right)$ has a negative value for critical types $\hat{\theta}_1(\hat{x}_0)$ and $\hat{\theta}_2(\hat{x}_0)$. That is, even the ideal custody arrangement $\hat{x}_0$ does not permit efficient arbitration.[13]

[13] Here is another way to see this: $\hat{\theta}_1(\hat{x}_0)$ is defined as the type of parent i that minimizes $\pi\left(\hat{x}_0,\hat{\theta}_1,\hat{\theta}_2\right)$. If minimization of π also translates into minimization of $\bar{\pi}\left(\hat{\theta}_1,\hat{\theta}_2(\hat{x}_0)\right)$, then it means that $\bar{\pi}\left(\hat{\theta}_1(\hat{x}_0),\hat{\theta}_2(\hat{x}_0)\right)$

To identify a sufficient condition for an efficient arbitration mechanism not to exist, we restrict attention to the following specification of our model: A parent's direct utility from custody depends only on the difference between the implemented custody arrangement and this parent's most desired arrangement. In particular we assume that $v_1(x, \theta_1) = v(x - \theta_1)$ and $v_2(x, \theta_2) = v(\theta_2 - x)$, where v is a strictly concave and twice continuously differentiable function. Under the interpretation that x is the proportion of time that the child spends with parent 1 according to the custody arrangement, these utility functions correspond to the two parents being symmetric in how they perceive custody arrangements giving them a lower (or higher) share than their preferred arrangement. See Figure 2 for an example where each parent's utility is decreasing faster for lower-than-preferred custody shares in comparison to higher-than-preferred shares. The symmetry between the parents implies that the ex-post efficient custody is the average of their preferred custody levels, i.e., $x^*(\theta_1, \theta_2) = \frac{\theta_1+\theta_2}{2}$. Notice that the quadratic-disutility function that we studied in the previous section is a special case for this specification.

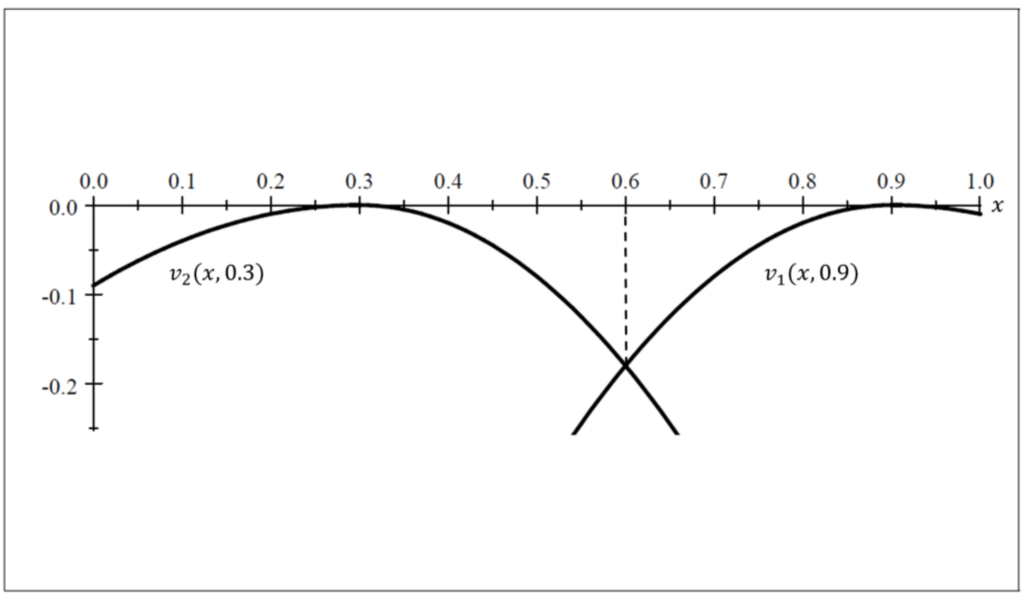

Figure 2: Parent 1 prefers $\theta_1 = 90\%$ share of the custody. Parent 2 prefers $1 - \theta_2 = 70\%$ share. The ex-post efficient custody is a 60% - 40% split.

Proposition 5 *If the second derivative of function v is strictly monotonic and convex, then*

takes a value lower than zero.

there is no default custody permitting efficient arbitration.

As is the case for the possibility result that we proved for quadratic disutility functions, the impossibility result holds regardless of the distribution functions of the two parents. As an example satisfying the conditions listed in the proposition, take $v(a) = -a^2 + \varepsilon a^3$ where $0 < \varepsilon < 1/3$. The first term in the direct utility functions of the parents is representing the quadratic disutility for the deviations from their preferred custody levels. And the second term can be considered as an adjustment term implying that receiving a higher weight than their preferred level in the custody arrangement is less costly than receiving a lower weight for the parents. In this example, the direct utility functions of the two parents and the maximized surplus function are as below:

$$\begin{aligned} v_1(x, \theta_1) &= -(x-\theta_1)^2 + \varepsilon (x-\theta_1)^3 \\ v_2(x, \theta_2) &= -(x-\theta_2)^2 - \varepsilon (x-\theta_2)^3 \\ S(\theta_1, \theta_2) &= -2\left(\frac{\theta_2-\theta_1}{2}\right)^2 + 2\varepsilon \left(\frac{\theta_2-\theta_1}{2}\right)^3 \end{aligned}$$

Proof. Suppose v'' is increasing and convex. The first property implies that v' is strictly convex. We also know from strict concavity of the utility function that v' is decreasing.

We look for an interior saddle point for function $\bar{\pi}\left(\hat{\theta}_1, \hat{\theta}_2\right)$. At such a saddle point, the first-order conditions imply that the first derivative of the function with respect to $\hat{\theta}_1$ equals zero:

$$E_{\theta_2} \frac{\partial v_1\left(x^*\left(\hat{\theta}_1, \theta_2\right), \hat{\theta}_1\right)}{\partial \theta_1} - \frac{\partial v_1\left(x^*\left(\hat{\theta}_1, \hat{\theta}_2\right), \hat{\theta}_1\right)}{\partial \theta_1} = -E_{\theta_2} v'\left(\frac{\theta_2 - \hat{\theta}_1}{2}\right) + v'\left(\frac{\hat{\theta}_2 - \hat{\theta}_1}{2}\right) = 0.$$

It follow from strict convexity of v' that

$$v'\left(\frac{E\theta_2 - \hat{\theta}_1}{2}\right) < v'\left(\frac{\hat{\theta}_2 - \hat{\theta}_1}{2}\right),$$

for any $\hat{\theta}_1, \hat{\theta}_2$ pair satisfying this first-order equation. Because v' is decreasing, this inequality implies that $E\theta_2 > \hat{\theta}_2$.

We now consider the second derivative of function $\bar{\pi}\left(\hat{\theta}_1, \hat{\theta}_2\right)$ with respect to $\hat{\theta}_1$ at the interior saddle point:

$$\frac{1}{2} E_{\theta_2} v''\left(\frac{\theta_2 - \hat{\theta}_1}{2}\right) - \frac{1}{2} v''\left(\frac{\hat{\theta}_2 - \hat{\theta}_1}{2}\right). \tag{2}$$

Notice that $E_{\theta_2} v''\left(\frac{\theta_2-\hat{\theta}_1}{2}\right) \geq v''\left(\frac{E\theta_2-\hat{\theta}_1}{2}\right)$ because v'' is convex. Moreover $v''\left(\frac{E\theta_2-\hat{\theta}_1}{2}\right) > v''\left(\frac{\hat{\theta}_2-\hat{\theta}_1}{2}\right)$ because v'' is increasing and $E\theta_2 > \hat{\theta}_2$. This establishes that the second derivative in (2) is positive when $\hat{\theta}_2$ is fixed at the critical type $\hat{\theta}_2(\hat{x}_0)$. Therefore, $\hat{\theta}_1(\hat{x}_0)$ uniquely satisfies the second-order sufficient conditions for minimization of $\bar{\pi}\left(\hat{\theta}_1, \hat{\theta}_2(\hat{x}_0)\right)$ as a function of $\hat{\theta}_1$. It follows from the discussion in the text that $\bar{\pi}\left(\hat{\theta}_1(\hat{x}_0), \hat{\theta}_2(\hat{x}_0)\right)$ takes a strictly negative value at these critical types and the ideal default custody $\hat{x}_0$ does not permit efficient arbitration. Because there is no efficient arbitration mechanism under the ideal default custody, there is no efficient mechanism under any default custody.

A similar proof can be constructed when v'' is decreasing and convex. In this case, v' is decreasing and strictly concave, implying that $\hat{\theta}_2 > E\theta_2$ at an interior saddle point. It again follows from monotonicity and convexity of v'' that $\hat{\theta}_1(\hat{x}_0)$ uniquely satisfies the second-order sufficient condition for minimization of $\bar{\pi}\left(\hat{\theta}_1, \hat{\theta}_2\right)$ given $\hat{\theta}_2 = \hat{\theta}_2(\hat{x}_0)$. ■

5 Concluding Remarks

In this paper, we study arbitration mechanisms that aim to allocate child custody in an ex-post efficient manner between the two parents with private preferences. Either parent can refuse the arbitration process, in favor of a court-ordered *default custody* arrangement. We show that the *ideal* default custody, which provides the best chances for efficient arbitration, is the custody arrangement that maximizes the sum of the reservation payoffs of the *critical* parent types, who are most reluctant to renounce this default custody. In addition to immediate implications, this observation leads to novel possibility and impossibility results on efficient arbitration. When both parents' payoffs are quadratically decreasing in the distance between their preferred custody regime and the implemented arrangement, efficient arbitration is possible only under the threat of reverting to the ideal default custody. For a more general class of preferences, which still depend on the difference between the preferred and implemented custody arrangements, we identify a sufficient condition under which there is no default custody permitting efficient arbitration. Both these results are general in the sense that they do not depend on the specifics of the parents' type distributions.

We provide brief discussions for alternative modeling choices.

Stochastic Default Custody

Suppose that, instead of committing to a single default custody arrangement, the legal system can support a randomization over different custody schemes. We can imagine a situation where the custody law leaves some discretion to the judge[14] and the system assigns the unsettled custody cases to judges with different preferences. In this case, the outside option of agreeing on an arbitration mechanism for the parents would be a draw from a distribution over different custody levels. Segal and Whinston (2011) show in a general setting that there exists a stochastic default arrangement which permits an efficient mechanism.[15] The default arrangement they use in their proof has the same distribution as the ex-post efficient custody $x^*(\theta_1, \theta_2)$, but does not depend on the realized values of θ_1 and θ_2. Under such a default custody regime, for each parent, rejecting arbitration would be essentially the same as accepting it and then reporting a random type with a distribution identical to the prior distribution of this parent's types. Therefore individual rationality follows from incentive compatibility.

The above default scheme mimicking the distribution of $x^*(\theta_1, \theta_2)$ permits efficient arbitration. However, it is not the ideal default custody arrangement that would give the highest revenue to a designer constrained to offer efficient mechanisms. Given the inherent risk aversion of the parents in our environment (reflected by their concave payoff functions), the ideal default custody rule would assign positive weight only to the extreme custody arrangements, where x_0 equals either 0 or 1. For instance, with quadratic disutility, the ideal default arrangement would be the one that sets $x_0 = 0$ with probability $\frac{E\theta_1 + E\theta_2 + 1}{4}$ and $x_0 = 1$ with the complementary probability.

Gradual Revelation Mechanisms

Considering the sequentiality of the real-life negotiations between the parents, some readers may be uncomfortable with our modeling of the arbitration process as a direct revelation mechanism, where each parent communicates with the mechanism just once and these communications take place simultaneously. A real-life arbitration can be such that first parent 1 reports her type, and parent 2 makes a report only after observing the first report. Alternatively, it could be that the parents do not report their types directly but they keep sending informative signals that lead to belief updates before the next round of communications. Such arbitration mechanisms, where information about the preferences are revealed gradually, would involve stronger constraints than the Bayesian incentive compatibility constraints we study in this paper. Nevertheless, it follows from Celik (2015) that, if the efficient custody arrangement can be supported by a simultaneous revelation mecha-

[14] Mnookin and Kornhauser (1979) give the "child's best interest" standard as an example to such a legal procedure.

[15] To be precise, Segal and Whinston show that, for *any* incentive-compatible allocation, there exists a stochastic default arrangement permitting implementation of that allocation.

nism, then the same arrangement can also be supported by a gradual revelation mechanism where the information is revealed in any arbitrary sequence. This is an implication of the monotonicity of the efficient custody arrangement in the types of the parents.

Ex-post Incentive Compatibility

An alternative strengthening of the incentive constraints would be replacing Bayesian versions of incentive compatibility and individual rationality with dominant-strategy incentive compatibility and ex-post individual rationality. Because the arbitration mechanism studied here is based on the Vickrey-Clarke-Groves mechanism, it is possible to modify the transfer functions to satisfy these more demanding conditions. However, this comes at the cost of replacing the ex-post budget constraint $t_1(\theta_1, \theta_2) + t_2(\theta_1, \theta_2) = 0$ with the ex-ante budget constraint $Et_1(\theta_1, \theta_2) + Et_2(\theta_1, \theta_2) = 0$.[16] This requires an enforcer of the arbitration mechanism who can act as a budget breaker with deep pockets. In real-life custody practice, there are some instances where the state jumps in to play the role of this budget breaker: In Quebec, the government takes over the child support payments when a parent defaults. In the US, states can retain child support to reimburse welfare payments.

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[16] See Mookherjee and Reichelstein (1992) for the construction of a dominant strategy mechanism which is equivalent to a Bayesian incentive compatible mechanism in the interim sense.

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