Bayesian social aggregation with almost-objective uncertainty*

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Abstract

We consider collective decisions under uncertainty, in which different agents may have not only different beliefs, but also different ambiguity attitudes —in particular, they may or may not be subjective expected utility maximizers. We assume that the space of possible states of nature is a Polish space. We consider sequences of acts which are "almost-objectively uncertain" in the sense that asymptotically, all agents almost-agree about the probabilities of the underlying events. We impose a weak ex ante Pareto axiom which applies only to asymptotic preferences along such almost-objective sequences. We show that this axiom implies that the social welfare function is utilitarian (i.e. a weighted sum of individual utility functions). But it does not impose any relationship between individual and collective beliefs, or between individual and collective ambiguity attitudes.

Keywords. Bayesian social aggregation; almost-objective uncertainty; generalized Hurwicz; second-order subjective expected utility; utilitarian.
JEL class: D70; D81.

1 Introduction

From a democratic point of view, collective decisions should be made by aggregating the preferences or opinions of the affected individuals. But almost all nontrivial decisions involve uncertainty. Normative decision theory considers the question of how rational agents should cope with such uncertainty. Bayesian social aggregation combines these two ingredients: it aims for collective decisions that are both rational and democratic. The foundational result is Harsanyi's (1955) Social Aggregation Theorem. Harsanyi considered a society in which all agents are von Neumann-Morgenstern (vNM) expected utility maximizers. He showed that if the vNM preferences of the social planner satisfy an ex ante

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Pareto axiom relative to the vNM preferences of the individuals, then the social welfare function —that is, the vNM utility function of the social planner —must be a weighted average of the individual vNM utility functions. Harsanyi interpreted this as a strong argument for utilitarianism.

Harsanyi's result is highly influential in social choice theory, but its dependence on the vNM framework curtails its applicability. The vNM framework assumes that all risks can be quantified with known, objective probabilities. But in many complex decision problems (e.g. macroeconomics, climate change, pandemics), it is not clear how to assign precise probabilities to the relevant contingencies. Indeed, when considering *sui generis* events in the future (e.g. hypothetical wars or financial crises in 2060), it is not clear that "objective" probabilities even exist. This led Savage (1954) to propose an approach to decision-making based on the maximization of *subjective* expected utility (SEU) —that is, expected utility computed using the agent's own "subjective" probabilistic beliefs.

A central tenet of the Savagean framework is that different rational agents may reasonably hold *different* subjective beliefs. But Mongin (1995) showed that Harsanyi's theorem breaks down in settings with heterogeneous beliefs. Mongin (1997) diagnosed the root of the problem in something he called *spurious unanimity*: different agents might have different utility functions *and* different beliefs, but these beliefs might "cancel out" to yield a unanimous ex ante preferences amongst the individuals for one act over another, thereby entailing (via the ex ante Pareto axiom) a corresponding ex ante social preference.

This suggests that to avoid Mongin's impossibility theorem, one should weaken the ex ante Pareto axiom to avoid cases of spurious unanimity. This strategy was realized in a landmark paper by Gilboa et al. (2004), who proposed a "restricted" ex ante Pareto axiom that only applied to acts for which all agents have the *same* probabilistic beliefs about the underlying events. Gilboa et al. showed that this restricted Pareto axiom has two consequences: (1) the social welfare function (SWF) must be a weighted sum of individual utility functions, and (2) the social beliefs must be a weighted average of individual beliefs.[1]

One objection to Gilboa et al.'s result is that it is not always appropriate to construct social beliefs as an arithmetic average of individual beliefs. For example, this way of aggregating beliefs does not interact well with Bayesian updating. In response, Dietrich (2021) has recently obtained a result similar to that of Gilboa et al. (2004), in which social beliefs are a weighted *geometric* average of individual beliefs. This ensures compatibility with Bayesian updating. But it does not address a broader issue. Different belief-aggregation rules are suitable in different contexts, and the criteria that determine the appropriate belief-aggregation rule are not necessarily the criteria that determine the correct social welfare function. The specification of collective beliefs is an *epistemic* problem, whereas the specification of the SWF is an *ethical* problem; there is no reason that these two problems should be solved by the same theorem, or even using the same data.[2] For this reason, Mongin and Pivato (2020) and Pivato (2021) have recently introduced weak Pareto axioms which entail a utilitarian SWF, but which do not impose *any* constraints on collective be-

[1] Brandl (2020) has recently obtained a similar result, but in his case, the SWF is *relative* utilitarian: it is a sum of the utility functions of individuals rescaled to range from 0 to 1. See also Billot and Qu (2021).

[2] See §4.6 of Pivato (2021) for further elaboration of these points.

liefs. They thus concentrate on the ethical problem, leaving the epistemic problem to be solved later by other methods. The present paper will take a similar approach.

All the results mentioned so far are vulnerable to another objection: they assume that all agents are expected utility maximizers. But in ambiguous decision environments, this might be inappropriate; it might be difficult to specify *any* single probability measure over contingencies as an adequate description of the uncertainty faced by an agent. This objection is both normative and descriptive. At a descriptive level, many agents might simply be *unable* to condense their uncertainty into a single probability measure. At a normative level, it is perhaps not even *rational* for an agent to resort to such a probabilistic description. These concerns have inspired a variety of *non-SEU* models of decision making. Typically such models represent an agent's beliefs not with a single probability measure but with an *ensemble* of probability measures, and in addition to her utility function, they often involve other parameters or mathematical structures that play a role in her decision process. For succinctness, we shall describe this entire package (i.e. a non-SEU decision model and its associated parameters and structures) as the agent's *ambiguity attitude*.

This raises the question of whether non-SEU ambiguity attitudes can be incorporated into collective decisions. But just as different agents can reasonably hold different probabilistic beliefs, different agents can reasonably adopt different ambiguity attitudes. Such heterogeneity leads once again to impossibility theorems (Chambers and Hayashi, 2006; Gajdos et al., 2008; Mongin and Pivato, 2015; Zuber, 2016). In general, to satisfy the ex ante Pareto axiom, all agents must not only have the same beliefs, but the same ambiguity attitudes —indeed, they must be SEU maximizers. Once again, to escape this undesirable conclusion, one must weaken the ex ante Pareto axiom; this strategy has been explored in a series of elegant papers by Alon and Gayer (2016), Danan et al. (2016), Qu (2017) and Hayashi and Lombardi (2019).[3] Like the foundational result of Gilboa et al. (2004), these more recent papers axiomatically characterize not only a SWF, but a procedure for aggregating individual beliefs into a collective belief. As already noted, non-SEU models generally represent agents' beliefs by ensembles of probability measures, so these procedures aggregate these ensembles. Thus, they are vulnerable to the same objections earlier raised against Gilboa et al. (2004) and Dietrich (2021): different belief-aggregation rules are appropriate in different environments, and in any case, collective beliefs should not necessarily be determined at the same time as the social welfare function. Furthermore, these theorems generally impose a particular ambiguity attitude on society (either in their hypotheses or in their conclusions).

Aside from heterogeneity of beliefs, another problem confronts the SEU framework adopted by Mongin (1995) and Gilboa et al. (2004): that of *state-dependent utility*. In certain situations, it may be perfectly reasonable for an agent's utility function to depend upon what state of nature is realized.[4] This creates two problems for Bayesian social aggregation. First, it makes it unclear how to impute probabilistic beliefs to the individual based on her ex ante preferences, as noted by Schervish et al. (1990) and Karni (1996), among others (see Baccelli (2017) for an excellent recent discussion of this problem). Second, in the

[3] See Mongin and Pivato (2016) or Fleurbaey (2018) for reviews of this literature.

[4] See e.g. Section 2.8 and Appendix 2A of Drèze (1987).

specification of the SWF, it raises the question of *which utility function* we should impute to each individual. For these reasons (among others) Duffie (2014) and Sprumont (2018, 2019) have rejected the approach pioneered by Gilboa et al. (2004) of weakening ex ante Pareto so as to separately aggregate beliefs and utilities. Instead Sprumont (2018, 2019) uses the full-strength ex ante Pareto axiom to characterize two approaches to Bayesian social aggregation based entirely the aggregation of individuals' ex ante preferences. The cost of these purely ex ante approaches is a loss of collective rationality: social decisions are no longer consistent with SEU maximization.[5]

The present paper develops an approach to collective decision-making under uncertainty that is compatible with both heterogeneity of beliefs *and* heterogeneity of risk-attitudes, and even compatible with certain forms of state-dependent utility. We exploit the concept of *almost-objective uncertainty* (due to Machina 2004, 2005) to formulate a weak Pareto axiom. We will show that this axiom is both necessary and sufficient for the social welfare function to be a weighted sum of individual utility functions. But it does not impose any relationship between individual and collective beliefs, or between individual and collective ambiguity attitudes. We see this as an advantage. Just as the specification of collective beliefs is an epistemic problem, the specification of collective ambiguity attitudes is a problem of *prudential rationality.* We feel that it is better to entirely separate these two problems from the ethical problem of specifying the SWF. We therefore focus exclusively on this last problem.

We will assume that the space of states of nature is a complete metric space (or more generally, a Polish space). This assumption is well-adapted to many practical decision problems, in which states of nature are vectors of real values ranging over some closed subset of a Euclidean space (or more generally, a Banach space). For example, in a financial decision problem, the state of nature would be a vector of prices. In social decisions related to climate change, the state of nature would be a vector of temperature, rainfall, insolation, and other meteorological and agronomic data.

The rest of this paper is organized as follows. Section 2 introduces the three classes of preferences we will consider in this paper: *subjective expected utility* (SEU), *generalized Hurwicz* (GH), and *second order subjective expected utility* (SOSEU). Section 3 introduces *almost-objective uncertainty*, and provides a versatile existence theorem for almost-objective uncertainty in Polish spaces (Proposition 1). Section 4 turns to Bayesian social aggregation, and contains our main results, which say that if all agents have SEU, GH, or SOSEU preferences and the social planner satisfies a weak Pareto axiom defined in terms of almost-objective uncertainty, then the social welfare function must be utilitarian (Theorems 1 and 2). These can be seen as analogies of Harsanyi's Social Aggregation Theorem that are robust against heterogeneity of subjective beliefs *and* ambiguity attitudes, as long as all agents' preferences belong to one of the three aforementioned classes.

Section 5 contains three results which are needed to prove the results of Section 4, but are also of independent interest; they describe the asymptotic behaviour of SEU, GH, or SOSEU representations in a situation of almost-objective uncertainty (Propositions 2, 3,

[5] See also Ceron and Vergopoulos (2019) for an interesting hybrid of ex ante and ex post approaches.

and 4). Finally, Section 6 extends some results from Sections 4 and 5 to a setting where agents have state-dependent utility functions. All proofs are in the Appendices.

2 Models of decision-making under uncertainty

Let $\mathcal{S}$ and $\mathcal{X}$ be measurable spaces —i.e. sets equipped with sigma-algebras.[6] We shall refer to $\mathcal{S}$ as the ***state space*** and $\mathcal{X}$ as the ***outcome space***. An ***act*** is a measurable function $\alpha : \mathcal{S} \longrightarrow \mathcal{X}$ that takes only finitely many values. Let $\mathcal{A}$ be the set of all acts. Let $\succeq$ be a preference order on $\mathcal{A}$. In the Savage model of uncertainty, $\mathcal{X}$ is a set of "outcomes", while $\mathcal{S}$ is a set of possible "states of nature"; the true state is unknown. The order $\succeq$ describes an agent's ex ante preferences. A ***representation*** of $\succeq$ is a function $V : \mathcal{A} \longrightarrow \mathbb{R}$ such that

$$\text{for all } \alpha, \beta \in \mathcal{A}, \quad \Big(\alpha \succeq \beta\Big) \quad \iff \quad \Big(V(\alpha) \geqslant V(\beta)\Big). \tag{1}$$

In particular, a ***subjective expected utility*** (SEU) ***representation*** for $\succeq$ consists of a probability measure[7] ρ on $\mathcal{S}$ and a bounded measurable function $u : \mathcal{X} \longrightarrow \mathbb{R}$ yielding a representation (1) in which

$$V(\alpha) \quad = \quad \int_{\mathcal{S}} u \circ \alpha \, \mathrm{d}\rho, \qquad \text{for all } \alpha \in \mathcal{A}. \tag{2}$$

Here, ρ is interpreted as the agent's *subjective beliefs* about the unknown state of nature, while u describes the utility she would obtain from each outcome. But as noted in Section 1, in situations of ambiguity, it might be inappropriate to represent an agent's beliefs as a single probability measure over $\mathcal{S}$. This has led to classes of preferences that use an *ensemble* of probability measures. In this paper, we shall consider two such classes.

Generalized Hurwicz representations. A representation V is called a ***generalized Hurwicz*** (GH) ***representation*** if there is a convex set $\mathcal{P}$ of probability measures over $\mathcal{S}$ and a bounded measurable utility function $u : \mathcal{X} \longrightarrow \mathbb{R}$, such that

$$\text{for all } \alpha \in \mathcal{A}, \quad \underline{V}(\alpha) \quad \leqslant \quad V(\alpha) \quad \leqslant \quad \overline{V}(\alpha), \tag{3}$$
$$\text{where} \quad \underline{V}(\alpha) \quad := \quad \inf_{\rho \in \mathcal{P}} \int_{\mathcal{S}} u \circ \alpha \, \mathrm{d}\rho \quad \text{and} \quad \overline{V}(\alpha) \quad := \quad \sup_{\rho \in \mathcal{P}} \int_{\mathcal{S}} u \circ \alpha \, \mathrm{d}\rho.$$

The idea here is that the agent is not only unsure of the true state of nature, but also unsure about the correct probability distribution to put on $\mathcal{S}$; the set $\mathcal{P}$ contains all probabilities that she considers *possible*. The GH representation (3) encompasses a very broad class of preferences. It reduces to the SEU representation (2) if $\mathcal{P}$ is a singleton. It obviously includes the class of *maximin SEU* (or *multiple priors*) preferences characterized by Gilboa and Schmeidler (1989) (for which $V(\alpha) = \underline{V}(\alpha)$, for all $\alpha \in \mathcal{A}$), and also the classical *Hurwicz* (or α-*maximin*) preferences introduced by Hurwicz (1951) and recently

[6] For simplicity, we shall not make these sigma-algebras explicit in our notation. A set will never be equipped with more than one sigma-algebra in this paper.

[7] All measures in this paper are *countably additive*, unless otherwise specified.

characterized by Hartmann (2021) (for which $V(\alpha) = q\,\underline{V}(\alpha) + (1-q)\,\overline{V}(\alpha)$, for all $\alpha \in \mathcal{A}$, for some constant $q \in [0,1]$).

In a setting where $\mathcal{X}$ is convex subset of a vector space (e.g. a simplex of probability measures), Cerreia-Vioglio et al. (2011) have introduced a class of *monotone, Bernoullian, Archimedean* (MBA) preferences, eponymously characterized by three mild axioms. In addition to SEU, maximin SEU, and Hurwicz preferences, the MBA class includes the *Choquet expected utility* preferences of Schmeidler (1989) and the *variational preferences* of Maccheroni et al. (2006). Any MBA preference admits a GH representation like (3) (Cerreia-Vioglio et al., 2011, Proposition 4). But in this representation, elements of $\mathcal{P}$ are *finitely* additive measures. Also, we will later assume that $\mathcal{S}$ is a metric space, whereas Cerreia-Vioglio et al. allow $\mathcal{S}$ to be any measurable space. On the other hand, they require $\mathcal{X}$ to be a convex set, whereas we allow $\mathcal{X}$ to be any measurable space. Thus, our framework does not exactly overlap with theirs. Nevertheless, their result suggests that the class of preferences admitting GH representations like (3) is quite extensive.[8]

Let $\mathcal{M}(\mathcal{S})$ be the vector space of all signed measures on $\mathcal{S}$. This becomes a Banach space when equipped with the total variation norm

$$\|\mu\|_{\mathrm{vr}} \quad := \quad \sup_{\substack{\mathcal{H}_1,\ldots,\mathcal{H}_N \subseteq \mathcal{S} \\ \text{disjoint Borel}}} \sum_{k=1}^{N} |\mu[\mathcal{H}_k]| \,. \tag{4}$$

We will say that a GH representation (3) is ***compact*** if the set $\mathcal{P}$ is compact in this norm.

SOSEU representations. Let $\mathcal{P}$ be a collection of probability measures on $\mathcal{S}$, equipped with the weak* topology. Then $\mathcal{P}$ itself is a measurable space when endowed with the Borel sigma algebra induced by this topology. Let $u : \mathcal{X} \longrightarrow \mathbb{R}$ be a bounded measurable function, let μ be a Borel probability measure on $\mathcal{P}$, and let $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ be a concave, increasing function. A ***second order subjective expected utility*** (SOSEU) representation is a representation of type (1) where

$$V(\alpha) \quad = \quad \int_{\mathcal{P}} \phi\left(\int_{\mathcal{S}} u \circ \alpha \;\mathrm{d}\rho\right) \,\mathrm{d}\mu[\rho], \qquad \text{for all } \alpha \in \mathcal{A}. \tag{5}$$

SOSEU representations have been axiomatically characterized by Klibanoff et al. (2005); see also Nau (2006), Seo (2009) and Ergin and Gul (2009). Like the GH representation (3), the SOSEU representation (5) describes an agent who is unsure about the correct probability distribution ρ to put on $\mathcal{S}$; the "second order probability distribution" μ encodes her beliefs about ρ. Meanwhile, ϕ encodes her attitudes towards ambiguity; in particular, the concavity of ϕ determines *ambiguity aversion.* If ϕ is linear, then (5) reduces to the SEU representation (2), with $\rho := \int_{\mathcal{P}} \rho' \;\mathrm{d}\mu[\rho']$. Likewise, if $\mathcal{P}$ is a singleton, then (5) reduces

[8] Social aggregation of MBA preferences has previously been studied by Herzberg (2013), Zuber (2016), and Danan et al. (2016); the first two papers concern impossibility theorems, while the third axiomatizes a social decision rule which simultaneously aggregates utilities and beliefs in this setting. Danan et al. (2016) refer to GH representations as *variable caution rules.*

to an SEU representation (2) (modulo the transformation ϕ, which does not change the agent's preferences in the SEU case).

In our main result, we shall assume that each agent has a preference over $\mathcal{A}$ with *either* a compact GH representation (3), *or* a SOSEU representation (5). Importantly, different agents might have *different* representations, with different choices of $\mathcal{P}$, q, μ and/or ϕ.

Contiguous representations. A given preference order on $\mathcal{A}$ may admit many different representations satisfying statement (1). For example, an SEU representation of type (2) is only unique up to positive affine transformations of the utility function u. The other representations described above have similarly qualified uniqueness properties. Thus, it is generally advisable to formulate axioms in terms of the preference order *itself*, rather than in terms of a particular representation. Nevertheless, our key axiom will be formulated in terms of representations. We shall now introduce a weak condition which guarantees that this axiom is independent of the choice of representation.

We shall say that a representation V of a preference order $\succeq$ is ***contiguous*** if its image $V(\mathcal{A})$ is an interval in $\mathbb{R}$. All of the representations introduced above are contiguous, under mild hypothesis. For example, if $\mathcal{X}$ is a connected topological space, and $u : \mathcal{X} \longrightarrow \mathbb{R}$ is continuous, then any representation of the form (2), (3) or (5) with u as its utility function is contiguous.[9] We will use the following fact, which is proved at the end of Appendix C.

If V_1 and V_2 are contiguous representations of the same preference order, then there is a continuous, strictly increasing function $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ such that $V_2 = \phi \circ V_1$. (6)

3 Almost-objective uncertainty

A ***measurable partition*** of $\mathcal{S}$ is a countable collection $\mathfrak{G} = \{\mathcal{G}_n\}_{n=1}^N$ (where $N \in \mathbb{N} \cup \{\infty\}$) of disjoint measurable subsets such that $\mathcal{S} = \bigsqcup_{n=1}^{N} \mathcal{G}_n$. For any $K \in \mathbb{N}$, let $\Delta^K := \{\mathbf{q} = (q_1, \ldots, q_K) \in \mathbb{R}_+^K;\ \sum_{k=1}^K q_k = 1\}$, the set of K-dimensional probability vectors.

Let $\mathcal{R}$ be a collection of probability measures on $\mathcal{S}$. Let $K \in \mathbb{N}$ and let $\mathbf{q} \in \Delta^K$. For all $n \in \mathbb{N}$, let $\mathfrak{G}^n := \{\mathcal{G}_1^n, \ldots, \mathcal{G}_K^n\}$ be a K-element measurable partition of $\mathcal{S}$. We shall say that the sequence of partitions $(\mathfrak{G}^n)_{n=1}^\infty$ is ***$\mathcal{R}$-almost-objectively uncertain*** and ***subordinate to*** $\mathbf{q}$ if, for all $\rho \in \mathcal{R}$, we have

$$\lim_{n\to\infty} \rho(\mathcal{G}_k^n) \quad = \quad q_k, \qquad \text{for all } k \in [1 \ldots K]. \tag{7}$$

For example, suppose $\mathcal{S} = [0,1]$, and let $\mathcal{R}$ be the set of all probability measures that are absolutely continuous with respect to the Lebesgue measure and whose density functions are continuous. Suppose $\mathbf{q} = (0.1, 0.2, 0.3, 0.4)$. For any number $s \in [0,1]$ and $n \in \mathbb{N}$, let $s_{(n)}$ be the nth digit in the decimal expansion of s.[10] For all $n \in \mathbb{N}$, let $\mathfrak{G}^n := \{\mathcal{G}_1^n, \mathcal{G}_2^n, \mathcal{G}_3^n, \mathcal{G}_4^n\}$,

[9] To see this, let α range over all constant-valued acts, to deduce that $V(\mathcal{A}) = u(\mathcal{X})$ or $V(\mathcal{A}) = \phi \circ u(\mathcal{X})$.

[10] There is a countable subset of elements of $[0,1]$ whose decimal expansions are not unique, so $s_{(n)}$ is not well-defined. But this set has Lebesgue measure zero, so it is irrelevant to this construction.

where $\mathcal{G}_1^n := \{s \in [0,1];\ \ s_{(n)} = 0\}$, $\mathcal{G}_2^n := \{s \in [0,1];\ \ s_{(n)} \in \{1,2\}\}$ $\mathcal{G}_3^n := \{s \in [0,1];\ s_{(n)} \in \{3,4,5\}\}$, and $\mathcal{G}_4^n := \{s \in [0,1];\ \ s_{(n)} \in \{6,7,8,9\}\}$. It is easily verified that $(\mathfrak{G}^n)_{n=1}^{\infty}$ is $\mathcal{R}$-almost-objectively uncertain and subordinate to $\mathbf{q}$.

Almost-objective uncertainty was first introduced by Poincaré (1912) to explain why it is reasonable to hold particular epistemic probabilities regarding a physical randomization device such as a roulette wheel, even if we do not have an exact understanding of how this apparent randomness is generated. Its first application to decision-making under ambiguity was due to Machina (2004, 2005), who also coined the term "almost-objective uncertainty". We shall apply it to the social aggregation of preferences under ambiguity.

Poincaré and Machina considered almost-objective uncertainty on the unit interval $[0,1]$, as in the above example. The first result of this paper will generalize this concept to a much broader collection of state spaces and probability measures. First we need some terminology. Recall that $\mathcal{M}(\mathcal{S})$ is the Banach space of signed measures on $\mathcal{S}$, with the total variation norm (4). A *closed subspace* of $\mathcal{M}(\mathcal{S})$ is a linear subspace $\mathcal{N} \subseteq \mathcal{M}(\mathcal{S})$ that is closed in the norm topology. If $\mathcal{H} \subseteq \mathcal{M}(\mathcal{S})$, then $\mathcal{H}$ *spans* $\mathcal{N}$ if $\mathcal{N}$ is the norm-closure of the vector space of all finite linear combinations of elements of $\mathcal{H}$. In this case, $\mathcal{N}$ is *separable* if $\mathcal{H}$ is countable.[11] We shall say that $\mathcal{N}$ is *nonatomic* if all elements of $\mathcal{N}$ are nonatomic. If $\mathcal{N}$ is spanned by $\mathcal{H}$, then this is equivalent to stipulating that all elements of $\mathcal{H}$ are nonatomic. We define $\langle\mathcal{N}\rangle := \{\mu \in \mathcal{M}(\mathcal{S});\ \ \mu$ is absolutely continuous with respect to some $\nu \in \mathcal{N}$, and the Radon-Nikodym derivative $\frac{d\mu}{d\nu}$ is bounded$\}$.[12]

Let $\mathcal{R}$ be some collection of probability measures on $\mathcal{S}$. We shall say that $\mathcal{R}$ is *tame* if there is a nonatomic, separable, closed linear subspace $\mathcal{N} \subseteq \mathcal{M}(\mathcal{S})$ such that $\mathcal{R} \subseteq \langle\mathcal{N}\rangle$. For example, $\mathcal{N}$ itself is tame. For another example, let $\mathcal{S} = [0,1]$ and let $\mathcal{R}$ be the set of all probability measures on $\mathcal{R}$ that are absolutely continuous with respect to Lebesgue, with density functions in $\mathcal{L}^{\infty}[0,1]$; then $\mathcal{R}$ is tame.

Our first result guarantees the existence of a rich family of almost-objectively uncertain partition sequences, for any tame family of probability measures. In this result, and the other main results of this paper, we assume that $\mathcal{S}$ is a *Polish space* —that is, a topological space homeomorphic to a complete, separable metric space —and we endow $\mathcal{S}$ with the Borel sigma-algebra.

Proposition 1 *Let $\mathcal{S}$ be a Polish space, and let $\mathcal{R}$ be a tame set of probability measures on $\mathcal{S}$. For any $K \in \mathbb{N}$ and $\mathbf{q} \in \Delta^K$, there is an $\mathcal{R}$-almost-objectively uncertain sequence of partitions $(\mathfrak{G}^n)_{n=1}^{\infty}$ subordinate to $\mathbf{q}$.*

4 Main results

As noted in Section 1, a central problem in Bayesian social aggregation is that different agents might have different probabilistic beliefs and different attitudes towards ambiguity. We shall now use almost-objective uncertainty to obviate these problems.

[11] This is equivalent to the topological definition of separability, i.e. that $\mathcal{N}$ has a countable dense subset.

[12] $\langle\mathcal{N}\rangle$ is a vector space, though not closed in the norm topology. But these facts are not relevant here.

Almost-objective acts. Let $\mathcal{R}$ be a collection of probability measures on $\mathcal{S}$. Let $\boldsymbol{\alpha} = (\alpha^n)_{n=1}^{\infty}$ be a sequence of acts. We shall say that $\boldsymbol{\alpha}$ is an *$\mathcal{R}$-almost-objective act* if there is a K-tuple of outcomes $\mathbf{x} \in \mathcal{X}^K$, and an $\mathcal{R}$-almost-objectively uncertain sequence of K-cell partitions $\boldsymbol{\mathcal{G}} = (\mathfrak{G}^n)_{n=1}^{\infty}$, with $\mathfrak{G}^n := \{\mathcal{G}_1^n, \ldots, \mathcal{G}_K^n\}$ for all $n \in \mathbb{N}$, such that for all $n \in \mathbb{N}$ and $k \in [1 \ldots K]$ we have $\alpha^n(s) = x_k$ for all $s \in \mathcal{G}_k^n$. If $\boldsymbol{\mathcal{G}}$ is subordinate to the probability vector $\mathbf{q} \in \Delta^K$, then we shall say that $\boldsymbol{\alpha}$ is *subordinate to* $(\mathbf{q}, \mathbf{x})$.

Let $\boldsymbol{\beta} = (\beta^n)_{n=1}^{\infty}$ be another almost-objective act. We shall say that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are *compatible* if β^n is also measurable with respect to $\mathfrak{G}^n$ for all $n \in \mathbb{N}$.

Asymptotic preferences. Let $\succeq$ be a preference order on $\mathcal{A}$. Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be two almost-objective acts. We shall say that $\succeq$ *asymptotically prefers* $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}$, and write $\boldsymbol{\alpha} \succ^{\infty} \boldsymbol{\beta}$, if there exists some contiguous representation V for $\succeq$, some $N \in \mathbb{N}$ and some $\epsilon > 0$ such that $V(\alpha^n) > V(\beta^n) + \epsilon$ for all $n \geqslant N$. In particular, this implies that $\alpha^n \succ \beta^n$ for all $n \geqslant N$, but it is a stronger requirement, because it incorporates an ϵ-sized "margin of error" in the superiority of $\boldsymbol{\alpha}$ over $\boldsymbol{\beta}$. Although this definition invokes a particular representation V, it is are independent of this representation, as follows:

If V_1 and V_2 are contiguous preferences for $\succeq$, and there exist $N_1 \in \mathbb{N}$ and $\epsilon_1 > 0$ such that $V_1(\alpha^n) > V_1(\beta^n) + \epsilon_1$ for all $n \geqslant N_1$, then there exist $N_2 \in \mathbb{N}$ and $\epsilon_2 > 0$ such that $V_2(\alpha^n) > V_2(\beta^n) + \epsilon_2$ for all $n \geqslant N_2$. (8)

Almost-objective Pareto. Let $\mathcal{I}$ be a set of individuals. Let o be another agent, representing a social planner or social observer. Let $\mathcal{J} = \mathcal{I} \sqcup \{o\}$. For all $j \in \mathcal{J}$, let $\succeq_j$ be a preference order on $\mathcal{A}$, with contiguous representation $V_j : \mathcal{A} \longrightarrow \mathbb{R}$. We shall require $\succeq_o$ to satisfy the following axiom, relative to $\{\succeq_i\}_{i \in \mathcal{I}}$ and $\mathcal{R}$:

Almost-objective Pareto. If $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are compatible $\mathcal{R}$-almost-objective acts, and $\boldsymbol{\alpha} \succ_i^{\infty} \boldsymbol{\beta}$ for all $i \in \mathcal{I}$, then $\boldsymbol{\alpha} \not\prec_o^{\infty} \boldsymbol{\beta}$.

This axiom does *not* require $\boldsymbol{\alpha} \succ_o^{\infty} \boldsymbol{\beta}$; it simply requires the social planner not to form the *opposite* asymptotic preference to that of the individuals. The axiom is vacuous unless $\mathcal{R}$-almost-objective acts exist. In our setting, existence will be ensured by Proposition 1.

Ex post Pareto. An act α is *riskless* if it is a constant function. Let us say that $\succeq_o$ satisfies the Ex post Pareto axiom with respect to $\{\succeq^i\}_{i \in \mathcal{I}}$ if, for any riskless $\alpha, \beta \in \mathcal{A}$,

- If $\alpha \succeq^i \beta$ for all $i \in \mathcal{I}$, then $\alpha \succeq \beta$.
- If, in addition, $\alpha \succ^i \beta$ for some $i \in \mathcal{I}$, then $\alpha \succ \beta$.

Minimal agreement and independent prospects. Suppose that each of the preference orders $\{\succeq_j\}_{j \in \mathcal{J}}$ has either a GH representation (3) or a SOSEU representation (5), with an associated utility function $u_j : \mathcal{X} \longrightarrow \mathbb{R}$. (In particular, some of $\{\succeq_j\}_{j \in \mathcal{J}}$ may have SEU representations like (2).) We shall say that the utility functions $\{u_i\}_{i \in \mathcal{I}}$ satisfy *Minimal Agreement* if there exist probability measures μ_1 and μ_2 on $\mathcal{X}$ such that

$\int_{\mathcal{X}} u_i \, \mathrm{d}\mu_1 > \int_{\mathcal{X}} u_i \, \mathrm{d}\mu_2$ for all $i \in \mathcal{I}$. In other words, there exist two "objective lotteries" over outcomes, for which all individuals have the same strict preference. We shall say that the collection $\{u_i\}_{i\in\mathcal{I}}$ satisfies ***Independent Prospects*** if, for all $j \in \mathcal{J}$, there exist outcomes $x, y \in \mathcal{X}$ such that $u_j(x) > u_j(y)$ whereas $u_i(x) = u_i(y)$ for all $i \in \mathcal{I}\backslash\{j\}$. Versions of these conditions are widespread in the literature on Bayesian social aggregation; see e.g. Mongin (1995, 1998), Alon and Gayer (2016), or Danan et al. (2016).

Utilitarianism and weak utilitarianism. Recall that u_o is the ex post utility function associated to the social preference order $\succeq_o$. We shall say that u_o is ***weakly utilitarian*** if there exist constants $c_i \geqslant 0$ for all $i \in \mathcal{I}$ and $b \in \mathbb{R}$ such that

$$u_o \quad = \quad b + \sum_{i\in\mathcal{I}} c_i \, u_i. \tag{9}$$

Here, it is possible that $c_i = 0$ for some $i \in \mathcal{I}$; thus, the preferences of some individuals might be ignored. If $c_i > 0$ for *all* $i \in \mathcal{I}$, then we say that u_o is ***utilitarian***. Suppose $\{u_i\}_{i\in\mathcal{I}}$ satisfy Independent Prospects. Then as shown in Appendix C, u_o is utilitarian if and only if it is weakly utilitarian and $\succeq$ satisfies Ex post Pareto with respect to $\{\succeq^i\}_{i\in\mathcal{I}}$. So our main focus will be on establishing *weak* utilitarianism. We now come to our main results.

Theorem 1 *Let $\mathcal{S}$ be a Polish space. Let $\mathcal{R}$ be a tame set of probability measures on $\mathcal{S}$. For all $j \in \mathcal{J}$, let $\succeq_j$ be a preference order on $\mathcal{A}$ admitting an SEU representation (2) with $\rho_j \in \mathcal{R}$. Assume that $\{u_i\}_{i\in\mathcal{I}}$ satisfy* Minimal Agreement*. Then $\succeq_o$ satisfies* Almost-objective Pareto *if and only if u_o is weakly utilitarian.*

In fact, Theorem 1 is a special case of the following result.

Theorem 2 *Let $\mathcal{S}$ be a Polish space. Let $\mathcal{R}$ be a tame set of probability measures on $\mathcal{S}$. For all $j \in \mathcal{J}$, let $\succeq_j$ be a preference order on $\mathcal{A}$, such that either*

- *$\succeq_j$ has a compact GH representation (3) with $\mathcal{P}_j \subseteq \mathcal{R}$; or*
- *$\succeq_j$ has a SOSEU representation (5) with $\mathcal{P}_j \subseteq \mathcal{R}$.*

Assume that $\{u_i\}_{i\in\mathcal{I}}$ satisfy Minimal Agreement*. Then $\succeq_o$ satisfies* Almost-objective Pareto *if and only if u_o is weakly utilitarian.*

5 Asymptotically objective expected utility

The proofs of Theorems 1 and 2 use the fact that the representations introduced in Section 2 take specific asymptotic values on almost-objective acts, as we now explain. Throughout this section, let $\mathcal{R}$ be a collection of probability measures on $\mathcal{S}$. Let $K \in \mathbb{N}$, let $\mathbf{q} \in \Delta^K$, let $\mathbf{x} \in \mathcal{X}^K$, and let $\boldsymbol{\alpha} = (\alpha^n)_{n=1}^{\infty}$ be an $\mathcal{R}$-almost-objective act subordinate to $(\mathbf{q}, \mathbf{x})$. Theorem 1 can be proved using the following result.

Proposition 2 *For any $\rho \in \mathcal{R}$, and any measurable $u : \mathcal{X} \longrightarrow \mathbb{R}$,*

$$\lim_{n\to\infty} \int_{\mathcal{S}} u \circ \alpha^n \, \mathrm{d}\rho \quad = \quad \sum_{k=1}^{K} q_k \, u(x_k).$$

Proposition 2 is a special case of the next two results, which are used to prove Theorem 2.

Proposition 3 *Let V be a compact GH representation (3) with $\mathcal{P} \subseteq \mathcal{R}$. Then*

$$\lim_{n\to\infty} V(\alpha^n) \quad = \quad \sum_{k=1}^{K} q_k \, u(x_k). \tag{10}$$

Proposition 4 *Let V be a SOSEU representation (5) with $\mathcal{P} \subseteq \mathcal{R}$. Then*

$$\lim_{n\to\infty} V(\alpha^n) \quad = \quad \phi\left(\sum_{k=1}^{K} q_k \, u(x_k)\right). \tag{11}$$

6 Extension to state-dependent utilities

As noted in Section 1, Bayesian social aggregation may encounter difficulties when individuals have state-dependent utilities.[13] The simplest version of state-dependent utility supposes that the agent has the same utility function in all states, up to some state-dependent scalar multiplier. In other words, the agent's state-dependent utility function $v : \mathcal{S} \times \mathcal{X} \longrightarrow \mathbb{R}$ has the form

$$v(s, x) \quad = \quad w(s)\, u(x), \quad \text{for all } s \in \mathcal{S} \text{ and } x \in \mathcal{X}, \tag{12}$$

where $u : \mathcal{X} \longrightarrow \mathbb{R}$ and $w : \mathcal{S} \longrightarrow \mathbb{R}_+$ are bounded measurable functions. Heuristically, u is an underlying state-*independent* utility function, while w assigns more "weight" to this utility in some states than in others. Let $\succeq$ be a preference on $\mathcal{A}$. Given a state-dependent utility function like (12), a *state-dependent SEU representation* is a representation (1) where

$$V(\alpha) \quad = \quad \int_{\mathcal{S}} v\left(s, \alpha(s)\right) \, \mathrm{d}\rho[s] \quad = \quad \int_{\mathcal{S}} w(s)\, u\left(\alpha(s)\right) \, \mathrm{d}\rho[s], \qquad \text{for all } \alpha \in \mathcal{A}. \tag{13}$$

Now let $\mathcal{R}$ be a collection of probability measures on $\mathcal{S}$. Let $K \in \mathbb{N}$, let $\mathbf{q} \in \Delta^K$, let $\mathbf{x} \in \mathcal{X}^K$, and let $\boldsymbol{\alpha} = (\alpha^n)_{n=1}^{\infty}$ be an $\mathcal{R}$-almost-objective act subordinate to $(\mathbf{q}, \mathbf{x})$. A straightforward modification of the proof of Proposition 2 yields the following result.

[13] One way reconcile the *ex ante* Pareto axiom with some form of social SEU maximization in an environment with heterogeneous beliefs is to introduce state-dependent *social welfare function*; see e.g. Mongin (1998, Prop.6), Chambers and Hayashi (2006, Thm.1), Desai et al. (2018, Thm.4), Sprumont (2019), and Mongin and Pivato (2020, Thm.1). But the issue under discussion here is state-dependent *individual* utility, not state-dependent *social* utility.

Proposition 5 *For any $\rho \in \mathcal{R}$, and any measurable $u : \mathcal{X} \longrightarrow \mathbb{R}$ and $w : \mathcal{X} \longrightarrow \mathbb{R}_+$,*

$$\lim_{n\to\infty} \int_{\mathcal{S}} w(s)\, u\left(\alpha(s)\right)\ \mathrm{d}\rho[s] \quad = \quad W(\rho) \sum_{k=1}^{K} q_k\, u(x_k), \quad \textit{where} \quad W(\rho) := \int_{\mathcal{S}} w\ \mathrm{d}\rho.$$

Using this, it is easy to prove the following state-dependent version of Theorem 1:

Theorem 3 *Let $\mathcal{S}$ be a Polish space. Let $\mathcal{R}$ be a tame set of probability measures on $\mathcal{S}$. For all $j \in \mathcal{J}$, let $\succeq_j$ be a preference order on $\mathcal{A}$ admitting a state-dependent SEU representation (13) for some $u_j : \mathcal{X} \longrightarrow \mathbb{R}$, $w_j : \mathcal{S} \longrightarrow \mathbb{R}_+$ and $\rho_j \in \mathcal{R}$. Assume that $\{u_i\}_{i\in\mathcal{I}}$ satisfy* Minimal Agreement*. Then $\succeq_o$ satisfies* Almost-objective Pareto *if and only if u_o is weakly utilitarian.*

In Theorem 3, it might seem surprising that the weight functions $\{w_j\}_{j\in\mathcal{J}}$ do not appear in the social welfare function. But as explained in Proposition 5, for all $j \in \mathcal{J}$, the weight function w_j and belief ρ_j effectively get collapsed into a constant $W_j(\rho_j) = \int_{\mathcal{S}} w_j\ \mathrm{d}\rho_j$. These constants then get absorbed into the weights in the weighted sum (9) defining the weakly utilitarian social welfare function.

GH representations like (3) and SOSEU representations like (5) also have state-dependent versions analogous to (13). But in these cases, the analogies of Proposition 5 are more complicated, because the weighting factor $W(\rho)$ varies as ρ ranges over $\mathcal{P}$. The resulting formulae not well-behaved enough to yield a state-dependent analogy to Theorem 2.

7 Discussion

We have considered a decision environment of radical uncertainty, in which the ex ante preferences of each agent admit either generalized Hurwicz representation or a second order subjective expected utility representation. We have introduced a very weak Pareto axiom, which applies only to asymptotic preferences along a sequence of acts for which all possible probabilistic beliefs entertained by all agents converge to the same limit. We have shown that social preferences satisfy this weak Pareto axiom if and only if the ex post social welfare function is a weighted sum of the ex post utility functions of the individuals. In other words, social preferences must be ex post utilitarian. Importantly, however, our results do not impose any relationship between collective beliefs and individual beliefs, or between collective ambiguity attitudes and individual ambiguity attitudes; for reasons already explained in Section 1, we see this as an advantage. We will now discuss the relationship between our results and the watershed paper of Gilboa et al. (2004).

Let $\mathfrak{G} = \{\mathcal{G}_1, \ldots, \mathcal{G}_K\}$ be a partition of $\mathcal{S}$. Let us say that $\mathfrak{G}$ is a *consensus partition* if there is some $\mathbf{q} \in \Delta^K$ such that $\rho_j(\mathcal{G}_k) = q_k$ for all $k \in [1 \ldots K]$ and $j \in \mathcal{J}$ —in other words, all agents *exactly agree* on the probabilities of all elements of $\mathfrak{G}$. If the measures $\{\rho_j\}_{j\in\mathcal{J}}$ are nonatomic, then the Dubins-Spanier Theorem says that such consensus partitions exist for any $\mathbf{q} \in \Delta^K$. Gilboa et al. (2004) proposed the following axiom:

Restricted Pareto. For any acts $\alpha, \beta \in \mathcal{A}$, if α and β are both measurable with respect to some consensus partition $\mathfrak{G}$, and $\alpha \succeq_i \beta$ for all $i \in \mathcal{I}$, then $\alpha \succeq_o \beta$.

This seems quite similar to Almost-objective Pareto. Indeed, if $\mathfrak{G}$ is a consensus partition, and we define $\mathfrak{G}_n := \mathfrak{G}$ for all $n \in \mathbb{N}$, then the sequence $(\mathfrak{G}_n)_{n=1}^{\infty}$ is trivially an "almost-objective" sequence with respect to the family $\{\rho_j\}_{j\in\mathcal{J}}$. Thus, if α and β are measurable with respect to $\mathfrak{G}$, and we define $\alpha_n := \alpha$ and $\beta_n := \beta$ for all $n \in \mathbb{N}$, then trivially, the sequences $\boldsymbol{\alpha} = (\alpha_n)_{n=1}^{\infty}$ and $\boldsymbol{\beta} := (\beta_n)_{n=1}^{\infty}$ are compatible almost-objective acts. Thus, any unanimous preference which is admissible to as input to Restricted Pareto is also admissible to Almost-objective Pareto, except that our axiom accepts a larger variety of inputs, and yields a weaker conclusion. From this perspective, it might seem as though we have just deployed a lot of topological machinery to obtain a variation of a result that Gilboa et al. (2004) already achieved by a much simpler argument in an abstract measure space.

However, there are several important differences between Almost objective Pareto and Restricted Pareto. First to apply Restricted Pareto in a particular situation, we must be able to recognize consensus partitions, which requires precise knowledge of the measures $\{\rho_j\}_{j\in\mathcal{J}}$ —something which may be difficult to achieve in practice. In contrast, to apply Almost-objective Pareto, we need only know that $\{\rho_j\}_{j\in\mathcal{J}}$ belong to some broad family $\mathcal{R}$ of probability measures. It is possible to determine whether a partition sequence is $\mathcal{R}$-almost-objectively uncertain without knowing anything about $\{\rho_j\}_{j\in\mathcal{J}}$, and also possible to construct such partition sequences on demand (e.g. using the methods of Appendix A).

Second, as agents acquire more information and Bayes-update their beliefs, *different* partitions of $\mathcal{S}$ will become consensus partitions. Thus, the range of application of Restricted Pareto will shift as the information available to the agents changes. Mongin and Pivato (2020) show that this makes Restricted Pareto vulnerable to a kind of "spurious unanimity" phenomenon: different agents might "spuriously" assign the same probabilities to the cells of a partition because they receive different information. This can lead Restricted Pareto to make recommendations which are obviously incorrect in light of the aggregate information of the entire group. Mongin and Pivato refer to this as *complementary ignorance*. In contrast, Almost-objective Pareto is formulated relative to a family $\mathcal{R}$ of probability measures, independent of the agents' current beliefs or current information. So it is not vulnerable to complementary ignorance.

Third, an important difference between the theorem of Gilboa et al. (2004) and our theorems is that ours do not impose any relationship between social beliefs and individual beliefs. As explained in Section 1, this gives our results added flexibility —especially in decision environments where a linear aggregation of beliefs is inappropriate.

Finally, although the Dubins-Spanier Theorem yields consensus partitions for a *finite* collection of probabilities, it does not apply to *infinite* collections. Thus, there is nothing analogous to Restricted Pareto for GH preferences or SOSEU preferences, in which each agent's beliefs might be represented by an infinite set of probabilities. So it is not straightforward to extend the result of Gilboa et al. (2004) to such ambiguity attitudes.[14]

[14] So far, the most general results along these lines are those of Danan et al. (2016).

A Proofs from Section 3

The proof of Proposition 1 requires an auxiliary concept and four preliminary lemmas. Recall that in Proposition 1, $\mathcal{S}$ was assumed to be a Polish space equipped with the Borel sigma algebra. Therefore, without loss of generality in this appendix we will suppose that $\mathcal{S}$ has a metric d, and when necessary, we will further assume that this metric is separable and/or complete. For any $\mathcal{Y} \subseteq \mathcal{S}$, the ***diameter*** of $\mathcal{Y}$ is defined: $\text{diam}(\mathcal{Y}) := \sup_{s,t\in\mathcal{Y}} d(s,t)$. For any $\epsilon > 0$, an ϵ***-partition*** is a measurable partition $\mathfrak{Y} = \{\mathcal{Y}_n\}_{n=1}^N$ of $\mathcal{S}$ (for some $N \in \mathbb{N}\cup\{\infty\}$) such that if $\text{diam}(\mathcal{Y}_n) \leqslant \epsilon$ for all $k \in [1 \ldots N]$.

Lemma A.1 *Let $(\mathcal{S}, d)$ be any metric space. Then $(\mathcal{S}, d)$ is separable if and only if it admits an ϵ-partition for all $\epsilon > 0$.*

Proof: "$\Longrightarrow$" Let $\{s_n\}_{n=1}^\infty$ be a countable dense subset of $\mathcal{S}$. Let $\epsilon > 0$. For all $s \in \mathcal{S}$, let $\mathcal{B}(s, \epsilon)$ be the open ball of radius ϵ around s. Now for all $N \in \mathbb{N}$, define $\mathcal{Y}_N := \mathcal{B}(s_N, \epsilon)\setminus\bigcup_{n=1}^{N-1}\mathcal{B}(s_N, \epsilon)$. Then $\{\mathcal{Y}_n\}_{n=1}^\infty$ is an ϵ-partition of $\mathcal{S}$.

"$\Longleftarrow$" For all $m \in \mathbb{N}$, let $\mathfrak{Y}^m = \{\mathcal{Y}_n^m\}_{n=1}^\infty$ be a $(\frac{1}{m})$-partition. For all $(n, m) \in \mathbb{N}^2$, let $s_{n,m} \in \mathcal{Y}_n^m$. Then $\{s_{n,m}\}_{n,m=1}^\infty$ is a countable dense subset of $\mathcal{S}$. So $\mathcal{S}$ is separable. □

Let $\mathcal{P}$ be a collection of Borel probability measures on $\mathcal{S}$, let $K \in \mathbb{N}$, and let $\mathbf{q} = (q_1, \ldots, q_K) \in \Delta^K$. A $\mathbf{q}$***-Poincaré sequence*** for $\mathcal{P}$ is a sequence $\{(\mathfrak{G}^n, \mathfrak{Y}^n, \epsilon_n)\}_{n=1}^\infty$, where for all $n \in \mathbb{N}$, $\mathfrak{G}^n = \{\mathcal{G}_1^n, \ldots, \mathcal{G}_K^n\}$ is a K-element measurable partition of $\mathcal{S}$, $\epsilon_n > 0$ and $\mathfrak{Y}^n$ is an ϵ_n-partition, such that

- $\lim_{n\to\infty} \epsilon_n = 0$.
- For all $\rho \in \mathcal{P}$, there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$, all $k \in [1 \ldots K]$, and all $\mathcal{Y} \in \mathfrak{Y}^n$, $\rho[\mathcal{G}_k^n \cap \mathcal{Y}] = q_k\, \rho[\mathcal{Y}]$ (and thus, $\rho[\mathcal{G}_k^n] = q_k$).

Example. Let $\mathcal{S} := [0, 1)$. Let $\mathcal{P} := \{\lambda\}$ where λ is the Lebesgue measure. Let $\mathbf{q} = (\frac{1}{2}, \frac{1}{2})$. For all $n \in \mathbb{N}$, let $\epsilon := 1/2^n$ and let $\mathfrak{Y}^n := \{\mathcal{Y}_1^n, \ldots, \mathcal{Y}_{2^n}^n\}$ where $\mathcal{Y}_k^n := [\frac{k-1}{2^n}, \frac{k}{2^n})$ for all $k \in [1 \ldots 2^n]$. Finally, let $\mathfrak{G}^n := \{\mathcal{G}_1^n, \mathcal{G}_2^n\}$, where

$$\mathcal{G}_1^n \quad := \quad \bigcup_{\substack{k=1 \\ k \text{ odd}}}^{2^{n+1}-1} \mathcal{Y}_k^{n+1} \quad \text{and} \quad \mathcal{G}_2^n \quad := \quad \bigcup_{\substack{k=2 \\ k \text{ even}}}^{2^{n+1}} \mathcal{Y}_k^{n+1}.$$

Then $\{(\mathfrak{G}^n, \mathfrak{Y}^n, \epsilon_n)\}_{n=1}^\infty$ is a $(\frac{1}{2}, \frac{1}{2})$-Poincaré sequence for $\{\lambda\}$.

Lemma A.2 *Let $(\mathcal{S}, d)$ be any separable metric space. Let $\mathcal{H} \subseteq \mathcal{M}(\mathcal{S})$ be a countable collection of nonatomic signed measures on $\mathcal{S}$. Let $\mathcal{F}$ be the linear subspace of $\mathcal{M}(\mathcal{S})$ consisting of all finite linear combinations of elements of $\mathcal{H}$. Let $\mathcal{P} \subseteq \mathcal{F}$ be the set of all probability measures in $\mathcal{F}$. Then for all $K \in \mathbb{N}$ and all $\mathbf{q} \in \Delta^K$, $\mathcal{P}$ has a $\mathbf{q}$-Poincaré sequence.*

Proof: Suppose that $\mathcal{H} = \{\eta_n\}_{n=1}^{\infty}$. For all $n \in \mathbb{N}$, the Hahn-Jordan Decomposition Theorem says that $\eta_n = \eta_n^+ - \eta_n^-$, where η_n^+ and η_n^- are either zero or positive measures. They are nonatomic because η_n is nonatomic. Thus, by replacing $\{\eta_n\}_{n=1}^{\infty}$ with $\{\eta_n^{\pm}\}_{n=1}^{\infty}$ if necessary, we can assume without loss of generality that all elements of $\mathcal{H}$ are positive, nonatomic measures.

Let $\{\epsilon_n\}_{n=1}^{\infty}$ be a positive sequence with $\lim_{n\to\infty} \epsilon_n = 0$. For all $N \in \mathbb{N}$, Lemma A.1 says $\mathcal{S}$ has an ϵ_N-partition $\mathfrak{Y}^N$.

Claim 1: *For all $N \in \mathbb{N}$, and all $\mathcal{Y} \in \mathfrak{Y}^N$, there is a measurable partition $\{\mathcal{G}_1^{\mathcal{Y}}, \ldots, \mathcal{G}_K^{\mathcal{Y}}\}$ of $\mathcal{Y}$ such that $n \in [1 \ldots N]$, we have*

$$\eta_n(\mathcal{G}_k^{\mathcal{Y}}) \quad = \quad q_k \cdot \eta_n(\mathcal{Y}), \quad \textit{for all } k \in [1 \ldots K]. \tag{A1}$$

Proof: Let $n \in [1 \ldots N]$. If $\eta_n(\mathcal{Y}) = 0$, then the equations (A1) are trivially satisfied for *any* partition $\{\mathcal{G}_{\mathcal{Y}}^1, \ldots, \mathcal{G}_{\mathcal{Y}}^K\}$. So, let $\mathcal{N} := \{n \in [1 \ldots N];\ \ \eta_n(\mathcal{Y}) > 0\}$; it suffices to construct a partition satisfying the equations (A1) for all $n \in \mathcal{N}$. For all $n \in \mathcal{N}$, let $\widetilde{\eta}_n$ be the nonatomic probability measure on $\mathcal{Y}$ defined by setting $\widetilde{\eta}_n(\mathcal{U}) := \eta_n(\mathcal{U})/\eta_n(\mathcal{Y})$ for all measurable $\mathcal{U} \subseteq \mathcal{Y}$. Thus $\{\widetilde{\eta}_n\}_{n\in\mathcal{N}}$ is a finite collection of nonatomic probability measures, so the Dubins-Spanier Theorem yields a partition $\{\mathcal{G}_1^{\mathcal{Y}}, \ldots, \mathcal{G}_K^{\mathcal{Y}}\}$ of $\mathcal{Y}$ such that

$$\widetilde{\eta}_n(\mathcal{G}_k^{\mathcal{Y}}) \quad = \quad q_k \quad \text{for all } k \in [1 \ldots K] \text{ and } n \in \mathcal{N}. \tag{A2}$$

(Aliprantis and Border, 2006, Theorem 13.34, p.478). For all $n \in \mathcal{N}$, multiply both sides of equation (A2) by $\eta_n(\mathcal{Y})$ to obtain equation (A1). ◇ `Claim 1`

Fix $N \in \mathbb{N}$, and apply Claim 1 to all $\mathcal{Y} \in \mathfrak{Y}^N$. Observe that the sets in the collection $\{\mathcal{G}_k^{\mathcal{Y}};\ \mathcal{Y} \in \mathfrak{Y}^N \text{ and } k \in [1 \ldots K]\}$ are all disjoint. For all $k \in [1 \ldots K]$, define

$$\mathcal{G}_k^N \quad := \quad \bigsqcup_{\mathcal{Y}\in\mathfrak{Y}^N} \mathcal{G}_k^{\mathcal{Y}}. \tag{A3}$$

Then $\{\mathcal{G}_1^N, \ldots, \mathcal{G}_K^N\}$ is a measurable partition of $\mathcal{S}$: these sets are are disjoint, and

$$\bigsqcup_{k=1}^{K} \mathcal{G}_k^N \quad = \quad \bigsqcup_{k=1}^{K} \left(\bigsqcup_{\mathcal{Y}\in\mathfrak{Y}^N} \mathcal{G}_k^{\mathcal{Y}} \right) \quad = \quad \bigsqcup_{\mathcal{Y}\in\mathfrak{Y}^N} \left(\bigsqcup_{k=1}^{K} \mathcal{G}_k^{\mathcal{Y}} \right) \quad = \quad \bigsqcup_{\mathcal{Y}\in\mathfrak{Y}^N} \mathcal{Y} \quad = \quad \mathcal{S}.$$

Furthermore, for all $\mathcal{Y} \in \mathfrak{Y}^N$, we have $\mathcal{G}_k^N \cap \mathcal{Y} = \mathcal{G}_k^{\mathcal{Y}}$ for all $k \in [1 \ldots K]$; thus, for all $n \in [1 \ldots N]$,

$$\eta_n(\mathcal{G}_k^N \cap \mathcal{Y}) \quad = \quad \eta_n(\mathcal{G}_k^{\mathcal{Y}}) \quad \underset{(*)}{=} \quad q_k\, \eta_n(\mathcal{Y}), \tag{A4}$$

where $(*)$ is by equation (A1).

Now, let $\rho \in \mathcal{P}$. Then there exists some $N \in \mathbb{N}$ such that ρ is a linear combination of $\eta_1, \ldots, \eta_N$. Thus, for any $n \geqslant N$, ρ is also a linear combination of $\eta_1, \ldots, \eta_n$ (with zero coefficients for $\eta_{N+1}, \ldots, \eta_n$). Thus, for all $\mathcal{Y} \in \mathfrak{Y}^n$ and all $k \in [1 \ldots K]$, equation (A4) yields $\rho[\mathcal{G}_k^n \cap \mathcal{Y}] = q_k\, \rho[\mathcal{Y}]$, as desired. □

Lemma A.3 *Suppose $(\mathcal{S}, d)$ is a complete, separable metric space. Let $K \in \mathbb{N}$, let $\mathbf{q} \in \Delta^K$, Let $\mathcal{P}$ be a collection of probability measures on $\mathcal{S}$, and let $\{(\mathfrak{G}^n, \mathfrak{Y}^n, \epsilon_n)\}_{n=1}^{\infty}$ be a $\mathbf{q}$-Poincaré sequence for $\mathcal{P}$. Let $\mathcal{L}$ be the set of all probability measures on $\mathcal{S}$ that are absolutely continuous with respect to* some *element of $\mathcal{P}$, with bounded Radon-Nikodym derivative. Then $(\mathfrak{G}^n)_{n=1}^{\infty}$ is $\mathcal{L}$-almost-objectively uncertain and subordinate to $\mathbf{q}$.*

Proof: Let $\lambda \in \mathcal{L}$ and let $k \in [1 \dots K]$. We will show that

$$\lim_{n\to\infty} \lambda(\mathcal{G}_k^n) \quad = \quad q_k. \tag{A5}$$

There exists $\rho \in \mathcal{P}$ such that $\lambda \ll \rho$. Let $\phi := \dfrac{\mathrm{d}\lambda}{\mathrm{d}\rho}$ and $C := \sup_{s\in\mathcal{S}} \phi(s)$. Then $C < \infty$ by hypothesis. Fix $\epsilon > 0$. Since $\mathcal{S}$ is complete and separable, it is Polish, so Lusin's Theorem yields a compact subset $\mathcal{K} \subseteq \mathcal{S}$ such that $\phi_{1\mathcal{K}}$ is uniformly continuous on $\mathcal{K}$ and

$$\rho(\mathcal{K}^\complement) \quad < \quad \frac{\epsilon}{8\,C}. \tag{A6}$$

(Aliprantis and Border, 2006, Theorem 12.8, p.438). It follows that

$$\lambda[\mathcal{K}^\complement] \quad = \quad \int_{\mathcal{K}^\complement} \phi \, \mathrm{d}\rho \quad \underset{(*)}{\leqslant} \quad C \cdot \rho[\mathcal{K}^\complement] \quad \underset{(\dagger)}{\leqslant} \quad C \cdot \frac{\epsilon}{8\,C} \quad = \quad \frac{\epsilon}{8}, \tag{A7}$$

where $(*)$ is because $0 \leqslant \phi(s) \leqslant C$ for all $s \in \mathcal{S}$, and $(\dagger)$ is by inequality (A6). Since $\{(\mathfrak{G}^n, \mathfrak{Y}^n, \epsilon_n)\}_{n=1}^{\infty}$ is a Poincaré sequence for $\mathcal{P}$, there is some $N_1 \in \mathbb{N}$ such that for all $n \geqslant N_1$ and all $\mathcal{Y} \in \mathfrak{Y}^n$,

$$\rho[\mathcal{G}_k^n \cap \mathcal{Y}] \quad = \quad q_k\, \rho[\mathcal{Y}]. \tag{A8}$$

Claim 1: *For all $n \geqslant N_1$, $\displaystyle\sum_{\mathcal{Y}\in\mathfrak{Y}^n} \left|\rho[\mathcal{G}_k^n \cap \mathcal{Y} \cap \mathcal{K}] - q_k\, \rho[\mathcal{Y} \cap \mathcal{K}]\right| \leqslant \frac{\epsilon}{4\,C}$.*

Proof: Let $n \geqslant N$. For all $\mathcal{Y} \in \mathfrak{Y}^n$,

$$\begin{aligned}
&\left|\rho[\mathcal{G}_k^n \cap \mathcal{Y} \cap \mathcal{K}] - q_k\, \rho[\mathcal{Y} \cap \mathcal{K}]\right| \\
&= \left|\rho[\mathcal{G}_k^n \cap \mathcal{Y} \cap \mathcal{K}] - \rho[\mathcal{G}_k^n \cap \mathcal{Y}] + \rho[\mathcal{G}_k^n \cap \mathcal{Y}] - q_k\, \rho[\mathcal{Y} \cap \mathcal{K}]\right| \\
&\underset{(*)}{=} \left|\rho[\mathcal{G}_k^n \cap \mathcal{Y} \cap \mathcal{K}] - \rho[\mathcal{G}_k^n \cap \mathcal{Y}] + q_k\, \rho[\mathcal{Y}] - q_k\, \rho[\mathcal{Y} \cap \mathcal{K}]\right| \\
&= \left|\rho[\mathcal{G}_k^n \cap \mathcal{Y} \cap \mathcal{K}] - \rho[\mathcal{G}_k^n \cap \mathcal{Y}] + q_k \left(\rho[\mathcal{Y}] - \rho[\mathcal{Y} \cap \mathcal{K}]\right)\right| \\
&\leqslant \left|\rho[\mathcal{G}_k^n \cap \mathcal{Y}] - \rho[\mathcal{G}_k^n \cap \mathcal{Y} \cap \mathcal{K}]\right| + q_k \left|\rho[\mathcal{Y}] - \rho[\mathcal{Y} \cap \mathcal{K}]\right| \\
&= \rho[\mathcal{G}_k^n \cap \mathcal{Y} \cap \mathcal{K}^\complement] + q_k\, \rho[\mathcal{Y} \cap \mathcal{K}^\complement].
\end{aligned} \tag{A9}$$

Here, $(*)$ is by equation (A8). Thus,

$$\sum_{\mathcal{Y}\in\mathfrak{Y}^n} \left|\rho[\mathcal{G}_k^n \cap \mathcal{Y} \cap \mathcal{K}] - q_k\, \rho[\mathcal{Y} \cap \mathcal{K}]\right| \quad \underset{(\dagger)}{\leqslant} \quad \sum_{\mathcal{Y}\in\mathfrak{Y}^n} \left(\rho[\mathcal{G}_k^n \cap \mathcal{Y} \cap \mathcal{K}^\complement] + q_k\, \rho[\mathcal{Y} \cap \mathcal{K}^\complement]\right)$$

$$= \rho\left[\bigsqcup_{\mathcal{Y}\in\mathfrak{Y}^n} (\mathcal{G}_k^n \cap \mathcal{Y} \cap \mathcal{K}^{\complement})\right] + q_k\,\rho\left[\bigsqcup_{\mathcal{Y}\in\mathfrak{Y}^n} (\mathcal{Y} \cap \mathcal{K}^{\complement})\right]$$
$$= \rho\left[\mathcal{G}_k^n \cap \mathcal{K}^{\complement} \cap \bigsqcup_{\mathcal{Y}\in\mathfrak{Y}^n} \mathcal{Y}\right] + q_k\,\rho\left[\mathcal{K}^{\complement} \cap \bigsqcup_{\mathcal{Y}\in\mathfrak{Y}^n} \mathcal{Y}\right]$$
$$\underset{(*)}{=} \rho\left[\mathcal{G}_k^n \cap \mathcal{K}^{\complement}\right] + q_k\,\rho\left[\mathcal{K}^{\complement}\right] \underset{(\diamond)}{\leqslant} \frac{\epsilon}{8\,C} + \frac{\epsilon}{8\,C} = \frac{\epsilon}{4\,C},$$

as claimed. Here, (†) is by applying inequality (A9) to each $\mathcal{Y} \in \mathfrak{Y}^n$, $(*)$ is because $\bigsqcup_{\mathcal{Y}\in\mathfrak{Y}^n} \mathcal{Y} = \mathcal{S}$, and $(\diamond)$ is by inequality (A6). $\diamond$ `Claim 1`

Recall that $\phi_{\restriction\mathcal{K}}$ is uniformly continuous on $\mathcal{K}$. Thus, there exists some $\delta > 0$ such that, for all $s_1, s_2 \in \mathcal{K}$, if $d(s_1, s_2) \leqslant \delta$, then $|\phi(s_1) - \phi(s_2)| < \frac{\epsilon}{4}$. Find $N_2 \in \mathbb{N}$ such that $\epsilon_n \leqslant \delta$ for all $n \geqslant N_2$. Thus, if $n \geqslant N_2$ and $\mathcal{Y} \in \mathfrak{Y}^n$, then $\mathrm{diam}(\mathcal{Y}) \leqslant \epsilon_n \leqslant \delta$, so that for all $y_1, y_2 \in \mathcal{Y} \cap \mathcal{K}$ we have $|\phi(y_1) - \phi(y_2)| < \frac{\epsilon}{4}$. Thus, there is some $c_{\mathcal{Y}} \in \mathbb{R}_+$ such that $|\phi(y) - c_{\mathcal{Y}}| < \frac{\epsilon}{4}$ for all $y \in \mathcal{Y} \cap \mathcal{K}$. Thus, for all $n \geqslant N_2$,

$$\left|\lambda[\mathcal{Y} \cap \mathcal{K} \cap \mathcal{G}_k^n] - c_{\mathcal{Y}} \cdot \rho[\mathcal{Y} \cap \mathcal{K} \cap \mathcal{G}_k^n]\right| \underset{(*)}{=} \left|\int_{\mathcal{Y}\cap\mathcal{K}\cap\mathcal{G}_k^n} (\phi - c_{\mathcal{Y}})\ \mathrm{d}\rho\right|$$
$$\leqslant \int_{\mathcal{Y}\cap\mathcal{K}\cap\mathcal{G}_k^n} |\phi - c_{\mathcal{Y}}|\ \mathrm{d}\rho \leqslant \int_{\mathcal{Y}\cap\mathcal{K}\cap\mathcal{G}_k^n} \frac{\epsilon}{4}\ \mathrm{d}\rho = \frac{\epsilon}{4} \cdot \rho[\mathcal{Y} \cap \mathcal{K} \cap \mathcal{G}_k^n], \quad \text{(A10)}$$

where $(*)$ is because $\phi = \dfrac{\mathrm{d}\lambda}{\mathrm{d}\rho}$. By a very similar argument,

$$\left|\lambda[\mathcal{Y} \cap \mathcal{K}] - c_{\mathcal{Y}}\rho[\mathcal{Y} \cap \mathcal{K}]\right| \leqslant \frac{\epsilon}{4} \cdot \rho[\mathcal{Y} \cap \mathcal{K}], \qquad \text{for all } n \geqslant N_2. \quad \text{(A11)}$$

Now, for any $n \in \mathbb{N}$,

$$\lambda[\mathcal{G}_k^n \cap \mathcal{K}] - q_k\,\lambda[\mathcal{K}] \underset{(*)}{=} \sum_{\mathcal{Y}\in\mathfrak{Y}} \lambda[\mathcal{G}_k^n \cap \mathcal{K} \cap \mathcal{Y}] - q_k \sum_{\mathcal{Y}\in\mathfrak{Y}} \lambda[\mathcal{K} \cap \mathcal{Y}]$$
$$= \sum_{\mathcal{Y}\in\mathfrak{Y}} c_{\mathcal{Y}}\,\rho[\mathcal{G}_k^n \cap \mathcal{K} \cap \mathcal{Y}] - \sum_{\mathcal{Y}\in\mathfrak{Y}} c_{\mathcal{Y}}\,\rho[\mathcal{G}_k^n \cap \mathcal{K} \cap \mathcal{Y}] + \sum_{\mathcal{Y}\in\mathfrak{Y}} \lambda[\mathcal{G}_k^n \cap \mathcal{K} \cap \mathcal{Y}]$$
$$-q_k \sum_{\mathcal{Y}\in\mathfrak{Y}} \lambda[\mathcal{K} \cap \mathcal{Y}] + q_k \sum_{\mathcal{Y}\in\mathfrak{Y}} c_{\mathcal{Y}}\rho[\mathcal{K} \cap \mathcal{Y}] - q_k \sum_{\mathcal{Y}\in\mathfrak{Y}} c_{\mathcal{Y}}\rho[\mathcal{K} \cap \mathcal{Y}]$$
$$= \sum_{\mathcal{Y}\in\mathfrak{Y}} \Big(c_{\mathcal{Y}}\,\rho[\mathcal{G}_k^n \cap \mathcal{K} \cap \mathcal{Y}] - q_k\,c_{\mathcal{Y}}\,\rho[\mathcal{K} \cap \mathcal{Y}]\Big) + \sum_{\mathcal{Y}\in\mathfrak{Y}} \Big(\lambda[\mathcal{G}_k^n \cap \mathcal{K} \cap \mathcal{Y}] - c_{\mathcal{Y}}\,\rho[\mathcal{G}_k^n \cap \mathcal{K} \cap \mathcal{Y}]\Big)$$
$$-q_k \sum_{\mathcal{Y}\in\mathfrak{Y}} \Big(\lambda[\mathcal{K} \cap \mathcal{Y}] - c_{\mathcal{Y}}\rho[\mathcal{K} \cap \mathcal{Y}]\Big), \quad \text{(A12)}$$

where $(*)$ is because $\bigsqcup_{\mathcal{Y}\in\mathfrak{Y}^n} \mathcal{Y} = \mathcal{S}$. Now let $N_\epsilon := \max\{N_1, N_2\}$. Then for all $n \geqslant N_\epsilon$,

$$\left|\lambda[\mathcal{G}_k^n \cap \mathcal{K}] - q_k\,\lambda[\mathcal{K}]\right|$$

$$
\begin{aligned}
&\underset{(\diamond)}{\leqslant} \left|\sum_{\mathcal{Y}\in\mathfrak{Y}} c_{\mathcal{Y}} \left(\rho[\mathcal{G}_k^n \cap \mathcal{K} \cap \mathcal{Y}] - q_k\, \rho[\mathcal{K}\cap\mathcal{Y}]\right)\right| + \left|\sum_{\mathcal{Y}\in\mathfrak{Y}} \left(\lambda[\mathcal{G}_k^n \cap \mathcal{K} \cap \mathcal{Y}] - c_{\mathcal{Y}}\, \rho[\mathcal{G}_k^n \cap \mathcal{K} \cap \mathcal{Y}]\right)\right| \\
&\qquad + q_k \left|\sum_{\mathcal{Y}\in\mathfrak{Y}} \left(\lambda[\mathcal{K}\cap\mathcal{Y}] - c_{\mathcal{Y}}\rho[\mathcal{K}\cap\mathcal{Y}]\right)\right| \\
&\leqslant \sum_{\mathcal{Y}\in\mathfrak{Y}} c_{\mathcal{Y}} \left|\rho[\mathcal{G}_k^n \cap \mathcal{K} \cap \mathcal{Y}] - q_k\, \rho[\mathcal{K}\cap\mathcal{Y}]\right| + \sum_{\mathcal{Y}\in\mathfrak{Y}} \left|\lambda[\mathcal{G}_k^n \cap \mathcal{K} \cap \mathcal{Y}] - c_{\mathcal{Y}}\, \rho[\mathcal{G}_k^n \cap \mathcal{K} \cap \mathcal{Y}]\right| \\
&\qquad + q_k \sum_{\mathcal{Y}\in\mathfrak{Y}} \left|\lambda[\mathcal{K}\cap\mathcal{Y}] - c_{\mathcal{Y}}\rho[\mathcal{K}\cap\mathcal{Y}]\right| \\
&\underset{(*)}{\leqslant} C \sum_{\mathcal{Y}\in\mathfrak{Y}} \left|\rho[\mathcal{G}_k^n \cap \mathcal{K} \cap \mathcal{Y}] - q_k\, \rho[\mathcal{K}\cap\mathcal{Y}]\right| + \sum_{\mathcal{Y}\in\mathfrak{Y}} \frac{\epsilon}{4}\rho[\mathcal{G}_k^n \cap \mathcal{K} \cap \mathcal{Y}] + q_k \sum_{\mathcal{Y}\in\mathfrak{Y}} \frac{\epsilon}{4}\rho[\mathcal{K}\cap\mathcal{Y}] \\
&\underset{(\dagger)}{\leqslant} C\frac{\epsilon}{4C} + \frac{\epsilon}{4}\sum_{\mathcal{Y}\in\mathfrak{Y}} \rho[\mathcal{G}_k^n \cap \mathcal{K} \cap \mathcal{Y}] + \frac{\epsilon}{4}\sum_{\mathcal{Y}\in\mathfrak{Y}} \rho[\mathcal{K}\cap\mathcal{Y}] \\
&\leqslant \frac{\epsilon}{4} + \frac{\epsilon}{4}\rho[\mathcal{G}_k^n \cap \mathcal{K}] + \frac{\epsilon}{4}\rho[\mathcal{K}] \quad \leqslant \quad \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} \quad = \quad \frac{3\epsilon}{4}.
\end{aligned} \tag{A13}
$$

Here, $(\diamond)$ is by equation (A12), while $(*)$ is by inequalities (A10) and (A11). Finally, $(\dagger)$ is by Claim 1, and also uses the fact that $q_k \leqslant 1$. Thus, for all $n \geqslant N_\epsilon$, we have:

$$
\begin{aligned}
|\lambda[\mathcal{G}_k^n] - q_k| &= \left|\lambda[\mathcal{G}_k^n \cap \mathcal{K}^{\complement}] + \lambda[\mathcal{G}_k^n \cap \mathcal{K}] - q_k \left(\lambda[\mathcal{K}] + \lambda[\mathcal{K}^{\complement}]\right)\right| \\
&\leqslant \left|\lambda[\mathcal{G}_k^n \cap \mathcal{K}^{\complement}]\right| + \left|\lambda[\mathcal{G}_k^n \cap \mathcal{K}] - q_k\, \lambda[\mathcal{K}]\right| + \left|\lambda[\mathcal{K}^{\complement}]\right| \\
&\underset{(*)}{\leqslant} \frac{\epsilon}{8} + \left|\lambda[\mathcal{G}_k^n \cap \mathcal{K}] - q_k\, \lambda[\mathcal{K}]\right| + \frac{\epsilon}{8} \\
&\underset{(\dagger)}{\leqslant} \frac{\epsilon}{8} + \frac{3\epsilon}{4} + \frac{\epsilon}{8} \quad = \quad \epsilon.
\end{aligned}
$$

where $(*)$ is by two applications of inequality (A7), while $(\dagger)$ is by inequality (A13).

We can construct such an N_ϵ for any $\epsilon > 0$. This proves the limit (A5). □

Lemma A.4 *Let $\mathcal{S}$ be any measurable space, and let $\mathcal{L}$ be a collection of probability measures on $\mathcal{S}$. Let $\mathcal{R}$ be the convex closure of $\mathcal{L}$ in the total variation norm. Let $\mathbf{q} \in \Delta^K$. If a partition sequence $(\mathfrak{G}^n)_{n=1}^\infty$ is $\mathcal{L}$-almost-objectively uncertain and subordinate to $\mathbf{q}$, then $(\mathfrak{G}^n)_{n=1}^\infty$ is $\mathcal{R}$-almost-objectively uncertain and subordinate to $\mathbf{q}$.*

Proof: Let $\mathcal{R}_0$ be the convex hull of $\mathcal{L}$. If $(\mathfrak{G}^n)_{n=1}^\infty$ is $\mathcal{L}$-almost-objectively uncertain and subordinate to $\mathbf{q}$, then it is easily shown that $(\mathfrak{G}^n)_{n=1}^\infty$ is also $\mathcal{R}_0$-almost-objectively uncertain subordinate to $\mathbf{q}$.

For all $n \in \mathbb{N}$, suppose $\mathfrak{G}^n = \{\mathcal{G}_1^n, \ldots, \mathcal{G}_K^n\}$. Let $\rho \in \mathcal{R}$. Then there is a sequence $\{\rho_m\}_{m=1}^\infty$ in $\mathcal{R}_0$ such that $\lim_{k\to\infty} \|\rho_m - \rho\|_{\text{vr}} = 0$. For all $k \in [1 \ldots K]$, we must show that the limit (7) holds for ρ.

Let $\epsilon > 0$. There exists $m \in \mathbb{N}$, with $\|\rho_m - \rho\|_{\mathrm{vr}} < \frac{\epsilon}{2}$. This means that $|\rho_m(\mathcal{G}) - \rho(\mathcal{G})| < \epsilon/2$ for all measurable $\mathcal{G} \subseteq \mathcal{S}$. In particular,

$$|\rho(\mathcal{G}_k^n) - \rho_m(\mathcal{G}_k^n)| \quad < \quad \frac{\epsilon}{2}, \quad \text{for all } n \in \mathbb{N}, \text{ all } k \in [1 \dots K]. \tag{A14}$$

The limit (7) holds for ρ_m, so there exists some $N_\epsilon \in \mathbb{N}$ such that

$$|\rho_m(\mathcal{G}_k^n) - q_k| \quad < \quad \frac{\epsilon}{2} \quad \text{for all } k \in [1 \dots K] \text{ and all } n \geqslant N_\epsilon. \tag{A15}$$

Combining inequalities (A14) and (A15) yields $|\rho(\mathcal{G}_k^n) - q_k| < \epsilon$ for all $n \geqslant N_\epsilon$. We can obtain such an N_ϵ for any $\epsilon > 0$. We conclude that the limit (7) holds for ρ. □

Proof of Proposition 1. If $\mathcal{R}$ is a tame set of probability measures, then there is a nonatomic, separable, closed linear subspace $\mathcal{N} \subseteq \mathcal{M}(\mathcal{S})$ such that for all $\rho \in \mathcal{R}$, there is some $\nu \in \mathcal{N}$ such that $\rho \ll \nu$ and $\frac{\mathrm{d}\rho}{\mathrm{d}\nu}$ is bounded. Since $\mathcal{N}$ is separable, it is spanned by a countable subset $\mathcal{H}$; since $\mathcal{N}$ is nonatomic, all elements of $\mathcal{H}$ are nonatomic. Let $\mathcal{F}$ be the linear subspace of $\mathcal{M}(\mathcal{S})$ consisting of all finite linear combinations of elements from $\mathcal{H}$. Then $\mathcal{N}$ is the norm-closure of $\mathcal{F}$. Let $\mathcal{P}$ be the set of all probability measures in $\mathcal{F}$. Let $\mathcal{L}$ be the set of all probability measures on $\mathcal{S}$ that are absolutely continuous with respect to some element of $\mathcal{P}$, with bounded Radon-Nikodym derivative.

Claim 1: *$\mathcal{R}$ is contained in the norm-closure of $\mathcal{L}$.*

Proof: Let $\rho \in \mathcal{R}$. Find $\nu \in \mathcal{N}$ such that $\rho \ll \nu$ and $\phi := \frac{\mathrm{d}\rho}{\mathrm{d}\nu}$ is bounded. Since $\mathcal{N}$ is the norm-closure of $\mathcal{F}$, there exists a sequence $(\nu_n)_{n=1}^{\infty}$ in $\mathcal{F}$ converging to ν in norm. For all $n \in \mathbb{N}$, let $\widetilde{\lambda}_n \in \mathcal{M}(\mathcal{S})$ be the measure such that $\widetilde{\lambda}_n \ll \nu_n$ and $\frac{\mathrm{d}\widetilde{\lambda}_n}{\mathrm{d}\nu_n} = \phi$. Next, let $\lambda_n := \widetilde{\lambda}_n/\ell_n$, where $\ell_n := \widetilde{\lambda}_n(\mathcal{S})$. Then $\lambda_n \in \mathcal{L}$. (*Proof:* By construction, λ_n is a probability measure, and $\lambda_n \ll \nu_n$. Let $\pi_n := \nu_n/\nu_n(\mathcal{S})$; then $\pi_n \in \mathcal{P}$, $\lambda_n \ll \pi_n$, and $\frac{\mathrm{d}\lambda_n}{\mathrm{d}\pi_n}$ is a multiple of ϕ, hence bounded.) To prove the claim, it suffices to show that the sequence $\{\lambda_n\}_{n=1}^{\infty}$ converges to ρ in norm. For any $n \in \mathbb{N}$,

$$\|\rho - \lambda_n\|_{\mathrm{vr}} \quad \leqslant \quad \left\|\rho - \widetilde{\lambda}_n\right\|_{\mathrm{vr}} + \left\|\widetilde{\lambda}_n - \lambda_n\right\|_{\mathrm{vr}}. \tag{A16}$$

Now, for any measurable $\mathcal{U} \subseteq \mathcal{S}$,

$$\begin{aligned} \left|\rho(\mathcal{U}) - \widetilde{\lambda}_n(\mathcal{U})\right| \quad &\underset{(*)}{=} \quad \left|\int_{\mathcal{U}} \phi \, \mathrm{d}\nu - \int_{\mathcal{U}} \phi \, \mathrm{d}\nu_n\right| \quad = \quad \left|\int_{\mathcal{U}} \phi \, \mathrm{d}(\nu - \nu_n)\right| \\ &\leqslant \quad \|\phi\|_\infty \cdot |\nu(\mathcal{U}) - \nu_n(\mathcal{U})|, \end{aligned}$$

where $(*)$ is because $\frac{\mathrm{d}\rho}{\mathrm{d}\nu} = \phi = \frac{\mathrm{d}\widetilde{\lambda}_n}{\mathrm{d}\nu_n}$. Combining this inequality with defining formula (4), we deduce that $\left\|\rho - \widetilde{\lambda}_n\right\|_{\mathrm{vr}} \leqslant \|\phi\|_\infty \cdot \|\nu - \nu_n\|_{\mathrm{vr}} \xrightarrow[n\to\infty]{(\dagger)} 0$, where $(\dagger)$ is because ν_n converges to ν in norm by hypothesis. Thus,

$$\lim_{n\to\infty} \left\|\rho - \widetilde{\lambda}_n\right\|_{\mathrm{vr}} \quad = \quad 0. \tag{A17}$$

Meanwhile,

$$\begin{aligned}
\left\|\widetilde{\lambda}_n - \lambda_n\right\|_{\rm vr} &= \|\ell_n\lambda_n - \lambda_n\|_{\rm vr} = |1-\ell_n|\cdot\|\lambda_n\|_{\rm vr} = |1-\ell_n| \\
&= \left|\rho(\mathcal{S}) - \widetilde{\lambda}_n(\mathcal{S})\right| \underset{(*)}{=} \left|\int_{\mathcal{S}} \phi\ \mathrm{d}\nu - \int_{\mathcal{S}} \phi\ \mathrm{d}\nu_n\right| \\
&= \left|\int_{\mathcal{S}} \phi\ \mathrm{d}(\nu-\nu_n)\right| \leqslant \|\phi\|_\infty \cdot \|\nu-\nu_n\|_{\rm vr} \xrightarrow[n\to\infty]{(\dagger)} 0,
\end{aligned}$$

where again, $(*)$ is because $\frac{\mathrm{d}\rho}{\mathrm{d}\nu} = \phi = \frac{\mathrm{d}\widetilde{\lambda}_n}{\mathrm{d}\nu_n}$ and $(\dagger)$ is because ν_n converges to ν in norm. Thus,

$$\lim_{n\to\infty} \left\|\widetilde{\lambda}_n - \lambda_n\right\|_{\rm vr} = 0. \tag{A18}$$

Equations (A16), (A17) and (A18) yield $\lim_{n\to\infty} \|\rho - \lambda_n\|_{\rm vr} = 0$, as desired. ◇ `Claim 1`

Let $\mathbf{q} \in \Delta^K$. Since $\mathcal{S}$ is separable, Lemma A.2 says that $\mathcal{P}$ has a $\mathbf{q}$-Poincaré sequence $\{(\mathfrak{G}^n, \mathfrak{Y}^n, \epsilon_n)\}_{n=1}^\infty$. Then Lemma A.3 says that $(\mathfrak{G}^n)_{n=1}^\infty$ is $\mathcal{L}$-almost-objectively uncertain, subordinate to $\mathbf{q}$. Then Lemma A.4 and Claim 1 says that $(\mathfrak{G}^n)_{n=1}^\infty$ is $\mathcal{R}$-almost-objectively uncertain, subordinate to $\mathbf{q}$. □

B Proofs from Section 5

The proof of the results in Section 4 use results from Section 5, so we will prove those first.

Proof of Proposition 2. By the standing hypotheses of Section 5, there is an $\mathcal{R}$-almost-objectively uncertain partition sequence $\boldsymbol{\mathcal{G}} = (\mathfrak{G}^n)_{n=1}^\infty$ subordinate to the probability vector $\mathbf{q}$, and for all $n \in \mathbb{N}$, the act α^n is $\mathfrak{G}^n$-measurable. Suppose $\mathbf{q} = (q_1, \ldots, q_K) \in \Delta^K$ (for some $K \in \mathbb{N}$). For all $n \in \mathbb{N}$, write $\mathfrak{G}^n := \{\mathcal{G}_1^n, \ldots, \mathcal{G}_K^n\}$, such that the limit equations (7) hold. By hypothesis, there is a K-tuple $\mathbf{x} \in \mathcal{X}^K$ such that for all $n \in \mathbb{N}$, all $k \in [1 \ldots K]$, and all $s \in \mathcal{G}_k^n$, we have $\alpha^n(s) = x_k$. Thus, for any $\rho \in \mathcal{R}$,

$$\int_{\mathcal{S}} u \circ \alpha^n \ \mathrm{d}\rho = \sum_{k=1}^K u(x_k)\, \rho(\mathcal{G}_k^n).$$

$$\text{Thus,} \quad \lim_{n\to\infty} \int_{\mathcal{S}} u \circ \alpha^n \ \mathrm{d}\rho = \lim_{n\to\infty} \sum_{k=1}^K u(x_k)\, \rho(\mathcal{G}_k^n) = \sum_{k=1}^K u(x_k) \lim_{n\to\infty} \rho(\mathcal{G}_k^n) \underset{(*)}{=} \sum_{k=1}^K u(x_k)\, q_k,$$

where $(*)$ is by the limit equations (7). □

Proof of Proposition 3. Recall the notation of equation (3). We will first show that the limit equation (10) holds for $\underline{V}$ and $\overline{V}$, and then show that it holds for V itself.

Claim 1: $\lim_{n\to\infty} \underline{V}(\alpha^n) = \sum_{k=1}^{K} q_k\, u(x_k)$.

Proof: Let $B := \|u\|_\infty$. Then $B < \infty$, and the sequence $\{\underline{V}(\alpha^n)\}_{n=1}^\infty$ is bounded in the interval $[-B, B]$, so it has convergent subsequences. To prove the claim, it suffices to show that *every* convergent subsequence of $\{\underline{V}(\alpha^n)\}_{n=1}^\infty$ converges to $\sum_{k=1}^{K} q_k\, u(x_k)$.

So, let $\{n(\ell)\}_{\ell=1}^\infty$ be an increasing sequence in $\mathbb{N}$ such that the subsequence $\{\underline{V}(\alpha^{n(\ell)})\}_{\ell=1}^\infty$ converges to some limit V^*. We must show that $V^* = \sum_{k=1}^{K} q_k\, u(x_k)$. For all $\ell \in \mathbb{N}$, define the linear function $v_\ell : \Delta(\mathcal{S}) \longrightarrow \mathbb{R}$ by

$$v_\ell(\rho) \quad := \quad \int_{\mathcal{S}} u \circ \alpha^{n(\ell)}\, \mathrm{d}\rho, \qquad \text{for all } \rho \in \Delta(\mathcal{S}). \tag{B1}$$

This function is continuous in the norm topology, while $\mathcal{P}$ is closed in this topology. Thus,

$$\underline{V}(\alpha^{n(\ell)}) \quad = \quad \min_{\rho\in\mathcal{P}} v_\ell(\rho) \quad = \quad v_\ell(\rho_\ell), \tag{B2}$$

for some $\rho_\ell \in \mathcal{P}$. Furthermore, $\mathcal{P}$ is norm-compact. Thus, the sequence $\{\rho_\ell\}_{\ell=1}^\infty$ has a subsequence $\{\rho_{\ell_m}\}_{m=1}^\infty$ that converges to some limit point $\rho_* \in \mathcal{P}$ in the norm topology.

Let $\epsilon > 0$. There exists $M_1 \in \mathbb{N}$ such that, for all $m \geqslant M_1$, $\|\rho_{\ell_m} - \rho_*\|_{\mathrm{vr}} < \frac{\epsilon}{3B}$. Thus, for all $n \in \mathbb{N}$ and all $m \geqslant M_1$,

$$\begin{aligned}\left|\int_{\mathcal{S}} u\circ\alpha^n\, \mathrm{d}\rho_{\ell_m} - \int_{\mathcal{S}} u\circ\alpha^n\, \mathrm{d}\rho_*\right| \quad &= \quad \left|\int_{\mathcal{S}} u\circ\alpha^n\, \mathrm{d}(\rho_{\ell_m} - \rho_*)\right| \\ &\leqslant \quad \|u\circ\alpha^n\|_\infty \cdot \|\rho_{\ell_m} - \rho_*\| \quad < \quad B\cdot\frac{\epsilon}{3B} \quad = \quad \frac{\epsilon}{3}.\end{aligned} \tag{B3}$$

In particular, setting $n := n(\ell_m)$ in inequality (B3) and invoking equation (B1) yields

$$\left|v_{\ell_m}(\rho_{\ell_m}) - v_{\ell_m}(\rho_*)\right| \quad < \quad \frac{\epsilon}{3}. \tag{B4}$$

Next, substituting equation (B2) into inequality (B4) yields

$$\left|\underline{V}(\alpha^{n(\ell_m)}) - v_{\ell_m}(\rho_*)\right| \quad < \quad \frac{\epsilon}{3}. \tag{B5}$$

Meanwhile, $\rho_* \in \mathcal{R}$, so Proposition 2 implies that there is some $N \in \mathbb{N}$ such that,

$$\left|\int_{\mathcal{S}} u\circ\alpha^n\, \mathrm{d}\rho_* - \sum_{k=1}^{K} q_k\, u(x_k)\right| \quad < \quad \frac{\epsilon}{3} \qquad \text{for all } n \geqslant N. \tag{B6}$$

Since the sequence $\{n(\ell_m)\}_{m=1}^{\infty}$ is strictly increasing, there is some $M_2 \in \mathbb{N}$ such that $n(\ell_m) > N$ for all $m \geqslant M_2$. From this and inequality (B6), it follows that

$$\left| \int_{\mathcal{S}} u \circ \alpha^{n(\ell_m)} \, \mathrm{d}\rho_* - \sum_{k=1}^{K} q_k \, u(x_k) \right| \quad < \quad \frac{\epsilon}{3}, \qquad \text{for all } m \geqslant M_2. \tag{B7}$$

Using the defining equation (B1), we can rewrite inequality (B7) as follows:

$$\left| v_{\ell_m}(\rho_*) - \sum_{k=1}^{K} q_k \, u(x_k) \right| \quad < \quad \frac{\epsilon}{3}, \qquad \text{for all } m \geqslant M_2. \tag{B8}$$

Finally, by hypothesis, $\lim\limits_{\ell\to\infty} \underline{V}(\alpha^{n(\ell)}) = V^*$. So there is some $L \in \mathbb{N}$ such that

$$\left| V^* - \underline{V}(\alpha^{n(\ell)}) \right| \quad < \quad \frac{\epsilon}{3}, \qquad \text{for all } \ell \geqslant L. \tag{B9}$$

Since the sequence $\{\ell_m\}_{m=1}^{\infty}$ is strictly increasing, there is some $M_3 \in \mathbb{N}$ such that $\ell_m > L$ for all $m \geqslant M_3$. From this and inequality (B9), it follows that

$$\left| V^* - \underline{V}(\alpha^{n(\ell_m)}) \right| \quad < \quad \frac{\epsilon}{3}, \qquad \text{for all } m \geqslant M_3. \tag{B10}$$

Now let $M_\epsilon := \max\{M_1, M_2, M_3\}$. Then for all $m \geqslant M_\epsilon$, we have

$$\begin{aligned}
&\left| V^* - \sum_{k=1}^{K} q_k \, u(x_k) \right| \\
&\leqslant \quad \left| V^* - \underline{V}(\alpha^{n(\ell_m)}) \right| \; + \; \left| \underline{V}(\alpha^{n(\ell_m)}) - v_{\ell_m}(\rho_*) \right| \; + \; \left| v_{\ell_m}(\rho_*) - \sum_{k=1}^{K} q_k \, u(x_k) \right| \\
&\underset{(*)}{<} \quad \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \quad = \quad \epsilon,
\end{aligned}$$

where $(*)$ is by inequalities (B5), (B8), and (B10).

This argument works for any $\epsilon > 0$. Thus, $V^* = \sum_{k=1}^{K} q_k \, u(x_k)$, as desired. ◇ `Claim 1`

By an argument very similar to Claim 1 (replacing min with max), we can show that

$$\lim_{n\to\infty} \overline{V}(\alpha^n) \quad = \quad \sum_{k=1}^{K} q_k \, u(x_k). \tag{B11}$$

Combining inequality (3) with Claim 1 and equation (B11) yields equation (10), proving the theorem. □

Proof of Proposition 4. For all $n \in \mathbb{N}$, define the function $\Phi_n : \mathcal{P} \longrightarrow \mathbb{R}$ by setting

$$\Phi_n(\rho) \quad := \quad \phi\left(\int_{\mathcal{S}} u \circ \alpha^n \, \mathrm{d}\rho\right), \qquad \text{for all } \rho \in \mathcal{P}.$$

Now, $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is concave, hence continuous. For all $\rho \in \mathcal{P}$, Proposition 2 says that

$$\lim_{n\to\infty} \int_{\mathcal{S}} u \circ \alpha^n \, \mathrm{d}\rho \;=\; \sum_{k=1}^{K} q_k \, u(x_k), \quad \text{hence} \quad \lim_{n\to\infty} \Phi_n(\rho) \;=\; \phi\left(\sum_{k=1}^{K} q_k \, u(x_k)\right). \tag{B12}$$

Let $\underline{u} := \inf_{x \in \mathcal{X}} u(x)$ and $\overline{u} := \sup_{x \in \mathcal{X}} u(x)$; these are finite because u is bounded. For all $\rho \in \mathcal{P}$ and $n \in \mathbb{N}$,

$$\underline{u} \;\leqslant\; \int_{\mathcal{S}} u \circ \alpha^n \, \mathrm{d}\rho \;\leqslant\; \overline{u}, \quad \text{hence} \quad \phi(\underline{u}) \;\leqslant\; \Phi_n(\rho) \;\leqslant\; \phi(\overline{u}),$$

because ϕ is increasing. Thus, the sequence of functions $\{\Phi_n\}_{n=1}^{\infty}$ are all bounded between the constants $\phi(\underline{u})$ and $\phi(\overline{u})$. Meanwhile, equation (B12) says that the sequence $\{\Phi_n\}_{n=1}^{\infty}$ converges pointwise to the constant function with value $\phi\left(\sum_{k=1}^{K} q_k \, u(x_k)\right)$. Thus, the Lebesgue Dominated Convergence Theorem says that

$$\lim_{n\to\infty} \int_{\mathcal{P}} \Phi_n \, \mathrm{d}\mu \quad = \quad \int_{\mathcal{P}} \phi\left(\sum_{k=1}^{K} q_k \, u(x_k)\right) \, \mathrm{d}\mu \quad = \quad \phi\left(\sum_{k=1}^{K} q_k \, u(x_k)\right). \tag{B13}$$

However, equation (5) says $V(\alpha^n) = \int_{\mathcal{P}} \Phi_n \, \mathrm{d}\mu$ for all $n \in \mathbb{N}$. Thus, equation (B13) yields equation (11). □

C Proofs from Section 4

Let $\mathcal{U}$ be the Banach space of bounded, measurable, real-valued functions on $\mathcal{X}$, endowed with the norm $\|\cdot\|_\infty$ defined by $\|u\|_\infty := \sup_{x \in \mathcal{X}} |u(x)|$ for all $u \in \mathcal{U}$. We shall use the following lemma, which is a straightforward consequence of the Separating Hyperplane Theorem.

Lemma C.1 *Let $\{u_j\}_{j \in \mathcal{J}} \subset \mathcal{U}$, and suppose $\{u_i\}_{i \in \mathcal{I}}$ satisfy* Minimal Agreement. *Suppose there exists $z \in \mathcal{X}$ such that $u_j(z) = 0$ for all $j \in \mathcal{J}$. Let $\mathcal{C}$ be the convex cone in $\mathcal{U}$ spanned by $\{u_i\}_{i \in \mathcal{I}}$ and 0. If $u_o \notin \mathcal{C}$, then there exist finitely additive probability measures ν_1 and ν_2 on $\mathcal{X}$ such that*

$$\int_{\mathcal{X}} u_o \, \mathrm{d}\nu_1 \;<\; \int_{\mathcal{X}} u_o \, \mathrm{d}\nu_2, \quad \textit{while} \quad \int_{\mathcal{X}} u_i \, \mathrm{d}\nu_1 \;>\; \int_{\mathcal{X}} u_i \, \mathrm{d}\nu_2 \quad \textit{for all } i \in \mathcal{I}. \tag{C1}$$

Proof: (Pivato, 2021, Lemma A.2). □

Theorem 1 is a special case of Theorem 2, so it suffices to prove the latter.

Proof of Theorem 2. "$\Longrightarrow$" (by contradiction) Suppose $\succeq_o$ satisfies Almost-objective Pareto, but u_o is *not* weakly utilitarian. Let $z \in \mathcal{X}$.

Claim 1: *We can assume without loss of generality that $u_j(z) = 0$ for all $j \in \mathcal{J}$.*

Proof: Let $c_j := u_j(z)$, and then define $\widetilde{u}_j(x) := u_j(x) - c_j$ for all $x \in \mathcal{X}$. If $\succeq_j$ has a GH representation (3), then $\succeq_j$ also admits a GH representation where u_j is replaced by $\widetilde{u}_j$. On the other hand, if $\succeq_j$ has a SOSEU representation (5), then define $\widetilde{\phi}_j(r) := \widetilde{\phi}_j(r + c_j)$ for all $r \in \mathbb{R}$. Then $\succeq_j$ also admits a SOSEU representation where u_j is replaced by $\widetilde{u}_j$ and ϕ_j is replaced by $\widetilde{\phi}_j$. ◇ Claim 1

Let $\mathcal{C}$ be the closed, convex cone in $\mathcal{U}$ spanned by $\{u_i\}_{i\in\mathcal{I}}$ and 0. Then u_o is weakly utilitarian if and only if $u_o \in \mathcal{C}$. Thus, if u_o is *not* weakly utilitarian, then $u_o \notin \mathcal{C}$, in which case Lemma C.1 yields finitely additive probability measures ν_1 and ν_2 on $\mathcal{X}$ satisfying the inequalities (C1). For all $j \in \mathcal{J}$, let $\epsilon^j := \left|\int_{\mathcal{X}} u_j \, \mathrm{d}\nu_1 - \int_{\mathcal{X}} u_j \, \mathrm{d}\nu_2\right|$. Let

$$\epsilon \quad := \quad \frac{1}{5} \min_{j\in\mathcal{J}} \epsilon^j. \tag{C2}$$

Then $\epsilon > 0$. Inequalities (C1) and definition (C2) yield

$$\int_{\mathcal{X}} u_o \, \mathrm{d}\nu_2 - \int_{\mathcal{X}} u_o \, \mathrm{d}\nu_1 \quad > \quad 5\,\epsilon, \tag{C3}$$

$$\text{while} \quad \int_{\mathcal{X}} u_i \, \mathrm{d}\nu_1 - \int_{\mathcal{X}} u_i \, \mathrm{d}\nu_2 \quad > \quad 5\,\epsilon, \quad \text{for all } i \in \mathcal{I}. \tag{C4}$$

Let $R := \max\{\|u_j\|_\infty\}_{j\in\mathcal{J}}$; this value is finite because $\{u_j\}_{j\in\mathcal{J}}$ are bounded. Let $N := \lceil R/\epsilon \rceil + 1$; then $N\epsilon > R$, so the interval $[-N\,\epsilon,\ N\,\epsilon)$ contains the ranges of $\{u_j\}_{j\in\mathcal{J}}$. For all $j \in \mathcal{J}$ and all $n \in [-N \dots N]$, let $\mathcal{Y}_n^j := (u_j)^{-1}[n\,\epsilon, (n{+}1)\,\epsilon)$. Then $\mathfrak{Y}^j := \{\mathcal{Y}_n^j\}_{n=-N}^N$ is a measurable partition of $\mathcal{X}$. Let $\mathfrak{Y}$ be the common refining partition of $\{\mathfrak{Y}^j\}_{j\in\mathcal{J}}$. This is a measurable partition of $\mathcal{X}$. Suppose it has K cells, and write $\mathfrak{Y} = \{\mathcal{Y}_k\}_{k=1}^K$. For all $k \in [1..K]$, let $p_k^1 := \nu_1(\mathcal{Y}_k)$ and $p_k^2 := \nu_2(\mathcal{Y}_k)$. Then $\mathbf{p}^1 := (p_k^1)_{k=1}^K$ and $\mathbf{p}^2 := (p_k^2)_{k=1}^K$ are K-dimensional probability vectors. For all $k \in [1 \dots K]$, let $x_k \in \mathcal{Y}_k$.

Claim 2: *For all $j \in \mathcal{J}$,*

$$\left|\sum_{k=1}^K p_k^1\, u_j(x_k) - \int_{\mathcal{X}} u_j \, \mathrm{d}\nu_1\right| \;<\; \epsilon \quad \textit{and} \quad \left|\sum_{k=1}^K p_k^2\, u_j(x_k) - \int_{\mathcal{X}} u_j \, \mathrm{d}\nu_2\right| \;<\; \epsilon.$$

Proof: To prove the first inequality, note that

$$\left|\sum_{k=1}^K p_k^1\, u_j(x_k) - \int_{\mathcal{X}} u_j \, \mathrm{d}\nu_1\right| \quad = \quad \left|\sum_{k=1}^K \nu_1(\mathcal{Y}_k)\, u_j(x_k) - \sum_{k=1}^K \int_{\mathcal{Y}_k} u_j \, \mathrm{d}\nu_1\right|$$

$$= \left|\sum_{k=1}^K \left(\int_{\mathcal{Y}_k} u_j(x_k)\ \mathrm{d}\nu_1 - \int_{\mathcal{Y}_k} u_j\ \mathrm{d}\nu_1\right)\right| \;=\; \left|\sum_{k=1}^K \left(\int_{\mathcal{Y}_k} u_j(x_k) - u_j(y)\ \mathrm{d}\nu_1[y]\right)\right|$$
$$\leqslant \sum_{k=1}^K \int_{\mathcal{Y}_k} |u_j(x_k) - u_j(y)|\ \mathrm{d}\nu_1[y] \;\underset{(*)}{<}\; \sum_{k=1}^K \int_{\mathcal{Y}_k} \epsilon\ \mathrm{d}\nu_1 \;=\; \sum_{k=1}^K \epsilon\,\nu_1(\mathcal{Y}_k) \;=\; \epsilon,$$

as claimed. Here $(*)$ is because for all $k \in [1\ldots K]$, we have $x_k \in \mathcal{Y}_k$ while $n\,\epsilon \leqslant u_j(y) < (n+1)\,\epsilon)$ for all $y \in \mathcal{Y}_k$, so that $|u_j(x_k) - u_j(y)| < \epsilon$ for all $y \in \mathcal{Y}_k$. The proof of the second inequality is similar. $\diamond$ `Claim 2`

Combining inequalities (C3) and (C4) with Claim 2 yields

$$\sum_{k=1}^K p_k^2\, u_o(x_k) - \sum_{k=1}^K p_k^1\, u_o(x_k) \;>\; 3\,\epsilon, \tag{C5}$$

$$\text{while} \quad \sum_{k=1}^K p_k^1\, u_i(x_k) - \sum_{k=1}^K p_k^2\, u_i(x_k) \;>\; 3\,\epsilon, \quad \text{for all } i \in \mathcal{I}. \tag{C6}$$

Let $\mathbf{q} \in \Delta^{K\times K}$ be the probability vector defined by $q_{k,\ell} := p_k^1\, p_\ell^2$ for all $k, \ell \in [1\ldots K]$. Since $\mathcal{S}$ is Polish and $\mathcal{R}$ is tame, Proposition 1 yields an $\mathcal{R}$-almost-objectively uncertain partition sequence $(\mathfrak{G}^n)_{n=1}^\infty$ subordinate to $\mathbf{q}$. For all $n \in \mathbb{N}$, write $\mathfrak{G}^n = \{\mathcal{G}_{k,\ell}^n\}_{k,\ell=1}^K$, with

$$\lim_{n\to\infty} \rho(\mathcal{G}_{k,\ell}^n) \;=\; q_{k,\ell}, \quad \text{for all } \rho \in \mathcal{R} \text{ and } k, \ell \in [1\ldots K]. \tag{C7}$$

For all $n \in \mathbb{N}$, and $\ell, k \in [1\ldots K]$, define $\mathcal{G}_{k,*}^n := \mathcal{G}_{k,1}^n \cup \mathcal{G}_{k,2}^n \cup \cdots \cup \mathcal{G}_{k,K}^n$ and $\mathcal{G}_{*,\ell}^n := \mathcal{G}_{1,\ell}^n \cup \mathcal{G}_{2,\ell}^n \cup \cdots \cup \mathcal{G}_{K,\ell}^n$. Then the equation (C7) yields

$$\lim_{n\to\infty} \rho(\mathcal{G}_{k,*}^n) \;=\; p_k^1 \quad \text{and} \quad \lim_{n\to\infty} \rho(\mathcal{G}_{*,\ell}^n) \;=\; p_\ell^2, \quad \text{for all } \rho \in \mathcal{R}. \tag{C8}$$

For all $n \in \mathbb{N}$, define acts $\alpha^n, \beta^n : \mathcal{S} \longrightarrow \mathcal{X}$ as follows.

- For all $k \in [1\ldots K]$, let $\alpha^n(s) := x_k$ for all $s \in \mathcal{G}_{k,*}^n$.
- For all $\ell \in [1\ldots K]$, let $\beta^n(s) := x_\ell$ for all $s \in \mathcal{G}_{*,\ell}^n$.

Thus, $\boldsymbol{\alpha} = (\alpha^n)_{n=1}^\infty$ and $\boldsymbol{\beta} = (\beta^n)_{n=1}^\infty$ are $\mathcal{R}$-almost-objectively uncertain acts. They are compatible because for all $n \in \mathbb{N}$, α^n and β^n are both $\mathfrak{G}^n$-measurable. By construction and equations (C8), $\boldsymbol{\alpha}$ is subordinate to $(\mathbf{p}^1, \mathbf{x})$, while $\boldsymbol{\beta}$ is subordinate to $(\mathbf{p}^2, \mathbf{x})$.

Claim 3: $\boldsymbol{\alpha} \succ_i^\infty \boldsymbol{\beta}$ *for all* $i \in \mathcal{I}$.

Proof: For all $i \in \mathcal{I}$, the preference $\succeq_i$ has a representation $V_i : \mathcal{A} \longrightarrow \mathbb{R}$ that is either a compact GH representation (3) or a SOSEU representation (5), with $\mathcal{P}_i \subseteq \mathcal{R}$ in either case. We will deal with these two cases separately.

Case 1. If V_j is a compact GH representation, then Proposition 3 says that

$$\lim_{n\to\infty} V_i(\alpha^n) \;=\; \sum_{k=1}^{K} p_k^1\, u_i(x_k) \quad \text{and} \quad \lim_{n\to\infty} V_i(\beta^n) \;=\; \sum_{k=1}^{K} p_k^2\, u_i(x_k).$$

Thus, there exists $N \in \mathbb{N}$ such that

$$\left|V_i(\alpha^n) - \sum_{k=1}^{K} p_k^1\, u_i(x_k)\right| \;<\; \epsilon \;\text{ and }\; \left|V_i(\beta^n) - \sum_{k=1}^{K} p_k^2\, u_i(x_k)\right| \;<\; \epsilon, \quad \text{for all } n \geqslant N. \tag{C9}$$

Combining inequalities (C6) and (C9), we obtain $V_i(\alpha^n) - V_i(\beta^n) > \epsilon$, for all $n \geqslant N$. Thus, $\boldsymbol{\alpha} \succ_i^{\infty} \boldsymbol{\beta}$, as claimed.

Case 2. If V_i is a SOSEU representation, then Proposition 4 says that

$$\lim_{n\to\infty} V_i(\alpha^n) \;=\; \phi_i\left(\sum_{k=1}^{K} p_k^1\, u(x_k)\right) \quad \text{and} \quad \lim_{n\to\infty} V_i(\beta^n) \;=\; \phi_i\left(\sum_{k=1}^{K} p_k^2\, u(x_k)\right). \tag{C10}$$

Now, $\phi_i : \mathbb{R}\longrightarrow\mathbb{R}$ is concave, therefore continuous. It is also increasing, hence bijective. Let $\mathcal{W} \subset \mathbb{R}$ be a compact neighbourhood of $\sum_{k=1}^{K} p_k^1\, u(x_k)$ and $\sum_{k=1}^{K} p_k^2\, u(x_k)$, and let $\mathcal{Z} := \phi(\mathcal{W})$. Then $\phi_i : \mathcal{W}\longrightarrow\mathcal{Z}$ is a continuous bijection with compact domain, thus, a homeomorphism. Thus, the inverse function $\phi_i^{-1} : \mathcal{Z}\longrightarrow\mathcal{W}$ is also continous. In fact, $\mathcal{Z}$ is compact, so ϕ_i^{-1} is uniformly continuous. So there is some $\delta > 0$ such that

$$\text{for all } z_1, z_2 \in \mathcal{Z}, \qquad \Big(|z_1 - z_2| < \delta\Big) \quad\Longrightarrow\quad \Big(\left|\phi_i^{-1}(z_1) - \phi_i^{-1}(z_2)\right| < \epsilon\Big). \tag{C11}$$

Now, $\mathcal{Z}$ is a neighbourhood around $\phi_i\left(\sum_{k=1}^{K} p_k^1\, u(x_k)\right)$ and $\phi_i\left(\sum_{k=1}^{K} p_k^2\, u(x_k)\right)$. Thus,

$$\text{for all } z \in \mathcal{Z}, \quad \left(\left|z - \phi_i\left(\sum_{k=1}^{K} p_k^1\, u(x_k)\right)\right| < \delta\right) \quad\Longrightarrow\quad \left(\left|\phi_i^{-1}(z) - \sum_{k=1}^{K} p_k^1\, u(x_k)\right| < \epsilon\right) \tag{C12}$$

$$\text{and} \quad \left(\left|z - \phi_i\left(\sum_{k=1}^{K} p_k^2\, u(x_k)\right)\right| < \delta\right) \quad\Longrightarrow\quad \left(\left|\phi_i^{-1}(z) - \sum_{k=1}^{K} p_k^1\, u(x_k)\right| < \epsilon\right) \tag{C13}$$

Now, the statements (C10) yield some $N \in \mathbb{N}$ such that for all $n \geqslant N$, we have

$$\left|V_i(\alpha^n) - \phi_i\left(\sum_{k=1}^{K} p_k^1\, u(x_k)\right)\right| \;<\; \delta \quad\text{and}\quad \left|V_i(\beta^n) - \phi_i\left(\sum_{k=1}^{K} p_k^2\, u(x_k)\right)\right| \;<\; \delta. \tag{C14}$$

Combining statements (C12), (C13), and (C14), we obtain

$$\left|\phi_i^{-1}\left(V_i(\alpha^n)\right) - \sum_{k=1}^{K} p_k^1\, u_i(x_k)\right| \;<\; \epsilon \;\text{ and }\; \left|\phi_i^{-1}\left(V_i(\beta^n)\right) - \sum_{k=1}^{K} p_k^2\, u_i(x_k)\right| \;<\; \epsilon, \quad \text{for all } n \geqslant N. \tag{C15}$$

Combining inequalities (C6) and (C15), we obtain $\phi_i^{-1}\left(V_i(\alpha^n)\right) - \phi_i^{-1}\left(V_i(\beta^n)\right) > \epsilon$, for all $n \geqslant N$. Thus, the logical contrapositive of statement (C11) implies that $|V_i(\alpha^n) - V_i(\beta^n)| > \delta$, for all $n \geqslant N$. Since ϕ_i is increasing, this means $V_i(\alpha^n) - V_i(\beta^n) > \delta$, for all $n \geqslant N$. Thus, $\boldsymbol{\alpha} \succ_i^\infty \boldsymbol{\beta}$, as claimed. ◇ `Claim 3`

By an argument identical to Claim 3, but using inequality (C5) rather than (C6), it is easy to prove that $\boldsymbol{\alpha} \prec_o^\infty \boldsymbol{\beta}$. This, together with Claim 3, is a violation of Almost-objective Pareto. Contradiction. To avoid this contradiction, u_o must be weakly utilitarian.

"$\Longleftarrow$" (by contradiction) Suppose u_o is weakly utilitarian; thus, $u_o = \sum_{i \in \mathcal{I}} c_i\, u_i$ for some constants $c_i \geqslant 0$. Suppose Almost-objective Pareto is violated. Then there exist compatible almost-objective acts $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ such that $\boldsymbol{\alpha} \succ_i^\infty \boldsymbol{\beta}$ for all $i \in \mathcal{I}$, while $\boldsymbol{\alpha} \prec_o^\infty \boldsymbol{\beta}$. Thus, for all $i \in \mathcal{I}$, there is some $\epsilon_i > 0$ and some $N_i \in \mathbb{N}$ such that

$$V_i(\alpha^n) - V_i(\beta^n) \quad > \quad 2\,\epsilon_i, \quad \text{for all } n \geqslant N_i, \tag{C16}$$

whereas there is some $\epsilon_o > 0$ and some $N_o \in \mathbb{N}$ such that

$$V_o(\beta^n) - V_o(\alpha^n) \quad > \quad 2\,\epsilon_o, \qquad \text{for all } n \geqslant N_o. \tag{C17}$$

There exists $K \in \mathbb{N}$, $\mathbf{p} \in \Delta^K$, and $\mathbf{x} \in \mathcal{X}^K$ such that $\boldsymbol{\alpha}$ is subordinate to $(\mathbf{p}, \mathbf{x})$. Likewise, There exists $L \in \mathbb{N}$, $\mathbf{q} \in \Delta^L$, and $\mathbf{y} \in \mathcal{X}^L$ such that $\boldsymbol{\beta}$ is subordinate to $(\mathbf{q}, \mathbf{y})$.

Claim 4: *For all* $i \in \mathcal{I}$, $\displaystyle\sum_{k=1}^K p_k\, u_i(x_k) \;-\; \sum_{\ell=1}^L q_\ell\, u_i(y_\ell) \;>\; 0.$

Proof: For all $i \in \mathcal{I}$, $\succeq_i$ has a representation $V_i : \mathcal{A} \longrightarrow \mathbb{R}$ that is either a compact GH representation (3) or a SOSEU representation (5), with $\mathcal{P}_i \subseteq \mathcal{R}$ in either case. We will deal with these cases separately.

Case 1. If V_i is a GH representation, then follow the argument in *Case 1* of the proof of Claim 3 to obtain $M_i \in \mathbb{N}$ such that

$$\left| V_i(\alpha^m) - \sum_{k=1}^K p_k\, u_i(x_k) \right| \;<\; \epsilon_i \quad \text{and} \quad \left| V_i(\beta^m) - \sum_{\ell=1}^L q_\ell\, u_i(y_\ell) \right| \;<\; \epsilon_i, \quad \text{for all } m \geqslant M_i. \tag{C18}$$

Now let $n \geqslant \max\{N_i, M_i\}$, and combine (C16) and (C18) to get the claimed inequality.

Case 2. Suppose V_i is a SOSEU representation. Let $\mathcal{W}_i \subset \mathbb{R}$ be a compact neighbourhood of $\sum_{k=1}^K p_k\, u_i(x_k)$ and $\sum_{\ell=1}^L q_\ell\, u_i(y_\ell)$. The convex function ϕ_i is continuous, hence uniformly continuous when restricted to $\mathcal{W}_i$. So there is some $\delta_i > 0$ such that

$$\text{for all } w_1, w_2 \in \mathcal{W}_i, \quad \Big(|z_1 - z_2| < 2\,\delta_i\Big) \quad \Longrightarrow \quad \Big(|\phi_i(z_1) - \phi_i(z_2)| < 2\,\epsilon_i\Big). \tag{C19}$$

Combining inequality (C16) with the contrapositive of statement (C19), we get

$$\phi_i^{-1}\left[V_i(\alpha^n)\right] - \phi_i^{-1}\left[V_i(\beta^n)\right] \quad > \quad 2\,\delta_i, \qquad \text{for all } n \geqslant N. \tag{C20}$$

Proposition 4 implies that there is some $M_i \in \mathbb{N}$ such that for all $m \geqslant M_i$,

$$\left|\phi_i^{-1}\left[V_j(\alpha^m)\right] - \sum_{k=1}^{K} p_k\, u_j(x_k)\right| \;<\; \delta_i \quad \text{and} \quad \left|\phi_i^{-1}\left[V_j(\beta^m)\right] - \sum_{\ell=1}^{L} q_\ell\, u_j(y_\ell)\right| \;<\; \delta_i, \tag{C21}$$

Now let $n \geqslant \max\{N_i, M_i\}$, and combine the inequalities (C20) and (C21) to obtain the claimed inequality. ◇ Claim 4

By an argument similar to Claim 4, but using inequality (C17) rather than (C16), one can show that

$$\sum_{k=1}^{K} p_k\, u_o(x_k) \;-\; \sum_{\ell=1}^{L} q_\ell\, u_o(y_\ell) \quad < \quad 0. \tag{C22}$$

Now, $u_o = \sum_{i \in \mathcal{I}} c_i\, u_i$. Thus,

$$\begin{aligned}\sum_{k=1}^{K} p_k\, u_o(x_k) \;-\; \sum_{\ell=1}^{L} q_\ell\, u_o(y_\ell) \;&=\; \sum_{k=1}^{K} p_k \sum_{i \in \mathcal{I}} c_i\, u_i(x_k) \;-\; \sum_{\ell=1}^{L} q_\ell \sum_{i \in \mathcal{I}} c_i\, u_i(y_\ell) \\ &=\; \sum_{i \in \mathcal{I}} c_i \left(\sum_{k=1}^{K} p_k\, u_i(x_k) \;-\; \sum_{\ell=1}^{L} q_\ell\, u_i(y_\ell)\right).\end{aligned} \tag{C23}$$

But $c_i \geqslant 0$ for all $i \in \mathcal{I}$, so equation (C23), inequality (C22) and Claim 4 are logically inconsistent. To avoid this contradiction, Almost-objective Pareto must be satisfied. □

We finish with the proofs of two statements made in the text.

Proof of statement (6). Let V_1 and V_2 be contiguous representations of $\succcurlyeq$. Define $\phi : V_1(\mathcal{A}) \longrightarrow V_2(\mathcal{R})$ as follows: for all $r \in V_1(\mathcal{A})$, set $\phi(r) := V_2(\alpha)$ for some $\alpha \in \mathcal{A}$ such that $V_1(\alpha) = r$. Since V_1 and V_2 represent the same order $\succcurlyeq$, this is well-defined independent of the choice of α; for the same reason, ϕ is a strictly increasing function. But V_1 and V_2 are contiguous, so that $V_1(\mathcal{A})$ and $V_2(\mathcal{A})$ are intervals of $\mathbb{R}$. It follows that ϕ is continuous. □

Utilitarianism vs. weak utilitarianism By definition, if u_0 is utilitarian, then it is weakly utilitarian. We will just show that ex post Pareto is satisfied. Let α and β be two riskless acts such that $\alpha \succcurlyeq^i \beta$ for all i. Assume that $\alpha(s) = x$ and $\beta(s) = y$ for all states $s \in \mathcal{S}$. We will have $V^i(\alpha) = u^i(x)$ and $V^i(\beta) = u^i(y)$, for all $i \in \mathcal{J}$. Thus, with $u^i(x) \geqslant u^i(y)$ for all $i \in \mathcal{I}$ and $u^0 = b + \sum_{i \in \mathcal{I}} c_i u_i$ we have $u^0(x) \geqslant u^0(y)$. Furthermore, if there is $i \in \mathcal{I}$ such that $u^i(x) > u^i(y)$, since $c_i > 0$, we will obviously have $u^0(x) > u^0(y)$.

Conversly, if u^0 is weakly utilitarian, then for all $i \in \mathcal{I}$, there is $c_i \geqslant 0$ such that $u^0 = b + \sum_{i \in \mathcal{I}} c_i u^i$. Let $i \in \mathcal{I}$. To show that $c_i > 0$ let $x^i, y^i \in \mathcal{X}$ such that $u^i(x^i) > u^i(y^i)$

and $u^j(x^i) = u^j(y^i)$ for $j \neq i$; this exists by the hypothesis of Independent Prospects. Considering the riskless acts $\alpha^i(s) = x^i$ and $\beta^i(s) = y^i$, we have $V^j(\alpha^i) \geqslant V^j(\beta^i)$ for all $j \in \mathcal{I}$ and $V^i(\alpha^i) > V^i(\beta^i)$. By Ex post Pareto, we have $V^0(\alpha^i) > V^0(\beta^i)$. Thus, $u^0(x^i) - u^0(y^i) = c^i(u^i(x^i) - u^i(y^i)) > 0$. But since $(u^i(x^i) - u^i(y^i)) > 0$, we get $c_i > 0$.

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