Pollution versus Inequality: Tradeoffs for Fiscal Policy

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Work in Progress

Abstract

In this paper, we investigate the impact of redistribution and taxation on inequality and pollution. We build a two-sector Ramsey model with a green good, a polluting good, heterogeneous households with non-homothetic preferences, and a subsistence level of consumption for the polluting commodity. We find that under heterogeneous preferences, lump-sum transfers reduce inequality but harms the environment. In the same vein, increasing the environmental tax under a high level of subsistence consumption leads to lower inequalities when coupled with high redistribution, but increases pollution. Therefore, there may be a tradeoff between inequality reduction and pollution mitigation. Looking at the stability properties of the economy, we find that the level of subsistence consumption and the externality matter. This leaves room for taxation and redistribution to play a role in the stability of the equilibrium.

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[0]All errors are ours.

1 Introduction

Pollution mitigation and income inequality have become hot topics in today's debates. Both climate change and income inequality are tried to be tackled in order to (i) preventing temperatures to rise above 2°C and (ii) preventing inequality to rise too much. When it comes to reducing polluting emissions, taxation can be seen as an efficient tool regarding the internalization of the externality. This has first been studied by Pigou (1920) and Baumol (1972). Environmental taxation has two benefits: it corrects the inefficiency induced by the externality, and it provides revenue for the government. This leads to a broad use of environmental taxes. Yet, environmentel tax reform face some opposition in the society, for example the Yellow Vests movement in France. Overestimating their losses, people rejected the implementation of a carbon tax even is the tax revenue is directly redistributed towards them (Douenne & Fabre, (2019)). Indeed, environmental taxes are usually considered regressive, as low income households spend a higher share of their income on polluting goods and hence bear most of the tax burden (Douenne (2020), Grainger & Kolstad (2010)). In this case, reducing emissions goes in opposition with inequality reduction: this is the so-called equity-efficiency tradeoff. Understanding the distributional effect of environmental tax reforms is thus key when trying to reduce both emissions and inequality at the same time, while making environmental taxes socially acceptable.

Recycling the tax revenue can be used to reconcile both goals. Part of the environmental literature has been dedicated to the analysis of the role of revenue recycling in the progressivity of environmental tax reforms and the reduction in inequality. When it comes to taxation and revenue recycling, two effects go in opposite directions: the use-side and the source-side impacts (Goulder et al. (2019)). The former describes the impact of taxation on wages, capital and transfer incomes ; the latter the impact on purchases. Therefore, the impact on emissions and inequality depends on which effect dominates the other. A high impact on prices should decrease pollution, while lump-sum transfers may mitigate this impact and reduce inequality.

This paper aims at analyzing the impact of environmental taxation and revenue recycling on pollution. We build a two-sector Ramsey model with heterogeneous households and a susbistence level of consumption for the polluting commodity. A tax is set on polluting consumption, and its revenue is given back to households through lump-sum transfers. The production of the polluting good exerts an externality on the households' utilities. We analyze the impact of redistribution and environmental taxation both on environmental quality and inequalities. Under heterogeneous preferences, there exists a tradeoff between reducing emissions and inequality: reducing the latter harms environmental quality. Similarly, a high level of subsistence consumption leads to increase environmental damages when the environmental tax is increased. Finally, we study the dynamics of the model under a local analysis. Focusing on the neighborhood of the steady state, we show that the level of subsistence consumption matters and the externality matter in the stability properties of our model. Therefore, the tax level and redistribution rate are key to ensure the stability of the steady state.

Our paper is linked to three strands of the literature. First, a huge part of the environmental literature analyzes the link between redistribution and pollution. Being mainly empirical, mixed effects are found, as for example in Lin & Li (2011). Berthe and Elie (2015) gather this literature and explain the differences in models (both theoretically and empirically) leading to this ambiguous effect. From a more macroeconomic point of view, Oueslati (2015) shows that in a two-sector endogeneous growth model, lump-sum transfers have no impact on aggregate variables, and hence on pollution. From a theoretical point of view, Rausch & Schwartz (2016) find that heterogeneity and non-homotheticity of preferences matter when it comes to the impact of redistribution on aggregate variables.

A second strand of interest is the one linking environmental taxation and inequality. Scalera (1996) and Hofkes (2001) look at the long-term effect of environmental taxation of endogeneous growth models. A great part of this literature focuses on taxation and revenue recycling, showing the importance of the recycling scheme when it comes to make environmental taxes progressive. Klenert et al. (2018) show the importance of lump-sum transfers in order to reduce inequality.

Finally, our paper is linked to the literature concerned by the stability properties of models with an environmental component. Antoci et al. (2005) show that, in a growth model with two goods, consumption choices can lead to indeterminacy. In the same vein, Itaya (2008) analyzes the impact of environmental taxation on long-run growth. In a model with a representative agent and an environmental externality, he shows that the impact of taxation on growth depends on the indeterminacy of the balanced growth path.

Our contribution is twofold. First, we analyze both the impact of taxation and redistribution on pollution and inequality. Like part of the literature on redistribution and pollution we find that, assuming the worker spends a higher share of her income on the polluting good relatively to the capitalist, reducing inequality is harmful for the environment. Alike the literature on taxation finding a positive relationship between higher taxation and lower pollution (Bovenberg and de Mooij (1997)), we find that introducing a susbistence level of polluting consumption mitigates this result. Second, we analyze the stability properties of our model. To our knowledge, this is the first paper analyzing the stability properties of a Ramsey economy with both externalities and non-homothetic preferences.

The rest of the paper is organized as follows. In Section 2, we present our framework. Section 3 and 4 state the equilibrium and the steady-state of the economy. Section 5 studies the impact of a change in taxation and in redistribution of the tax revenue. In Section 6, we analyze the local stability properties of our economy. Section 7 provides concluding remarks. All proofs are relegated to the Appendix.

2 The Model

We consider a infinite-horizon two-sector model with environmental externalities. There are three types of agents: households, firms and a government. Households are infinitely-lived and heterogeneous in their discount factors. Firms produce either a clean good, or a polluting good that exerts an externality on household's utilities. Government's intervention, through taxation and/or redistribution, aims at reducing environmental damages as well as income and consumption inequalities.

2.1 Firms

There are 2 sectors in the economy, each composed of a representative firm: the clean sector produces Y_{gt} and the polluting sector produces Y_{pt} in period t. Each sector uses capital K_{jt} and labor N_{jt} $(j = p, g)$ to produce output according to a Cobb-Douglas production function:

$$Y_{jt} = A_j K_{jt}^{\eta} N_{jt}^{1-\eta} \tag{1}$$

with A_j a productivity parameter. We assume $A_g \neq A_p$.

In the polluting sector, output is composed of a consumption good c_{pt} only, while the clean sector produces a consumption good c_{gt} and a capital good K_t. The capital good includes immaterial and non-polluting inputs used in the production process, such as R&D investment and human capital.

Each firm j $(j = p, g)$ seeks to maximize its profit:

$$\pi_{jt} = p_{jt|j=g} Y_{jt} - r_t K_{jt} - w_t N_{jt}$$

with r_t the rental rate of capital and w_t the wage rate.
Assuming capital and labor are perfectly mobile across sectors, first-order conditions for profit maximization give:

$$r_t = \eta p_t A_g K_{gt}^{\eta-1} N_{gt}^{1-\eta} = \eta A_p K_{pt}^{\eta-1} N_{pt}^{1-\eta} \tag{2}$$

$$w_t = (1-\eta) p_t A_g K_{gt}^{\eta} N_{gt}^{-\eta} = (1-\eta) A_p K_{pt}^{\eta} N_{pt}^{-\eta} \tag{3}$$

2.2 Pollution

We consider pollution as a flow due to production in the polluting sector. One can think for example of gases emitted during the production process such as sulfur dioxide and carbon monoxide, which have a short lifetime and hence can be considered as flow pollutants (Liu & Liptak (2000)).

Pollution is given by:

$$E_t = \frac{1}{\gamma Y_{pt}} \tag{4}$$

with γ the emission factor.

2.3 The Government

The government intervenes in the economy for two reasons: mitigating pollution and reducing income inequalities. To reach those goals, two instruments are available in the economy: a tax on the polluting commodity, denoted τ and the redistribution rate, denoted ε.
The only tax revenue the government gets is through the tax on the polluting commodity, which is

used for redistribution towards both households. The government is subject to a budget balance rule:

$$T_t = T_{1t} + T_{2t} = \tau(c_{p1t} + c_{p2t}) = \tau c_{pt}, \tag{5}$$

with $T_{1t} = \varepsilon T_t$ and $T_{2t} = (1-\varepsilon)T_t$ the transfers given to household 1 and 2 respectively.

2.4 Households

The economy is composed of two infinitely-lived households ($i = \{1, 2\}$). At every period t, each household i consumes a clean good c_{git} at price p_t and a polluting good c_{pit} which is the numeraire and taxed at a rate τ. There exists a susistence consumption level of the polluting good, denoted c_0. This susbsistence level can be seen for example as the minimum level of energy a household needs to consume in order to live in a decent manner. Consumption can be summarized by a basket of good C_{it} purchased at price P_{it}.
Households also derive some utility from environmental quality, denoted E_t. Finally, they can save a_{it}, supply $n_i = \frac{1}{2}$ of labor at wage rate w_t, and receive a lump-sum transfer T_{it}.
Instantaneous preferences write

$$U_i(c_{pit}, c_{git}, E_t) = E_t^{\mu} \frac{((c_{pit} - c_0)^{\alpha} c_{git}^{1-\alpha})^{1-\sigma}}{1-\sigma}, \tag{6}$$

with $c_{pit}^{\alpha} c_{git}^{1-\alpha} = C_{it}$. Pollution is taken as given by households, such that it only plays the role of an externality in the utility function.

Assumption 1 $\sigma < 1$.

The budget constraint writes

$$c_{pit}(1+\tau) + c_{git}p_t + p_t(a_{it+1} - (1-\delta)a_{it}) = \frac{w_t}{2} + r_t a_{it} + T_{it}.$$

Households maximize their discounted lifetime utility $\sum_{t=0}^{\infty} \beta_i^t U_i(c_{pit}, c_{git}, E_t)$ with respect to their budget and borrowing constraints $a_{it} \geq 0$. β_i is the discount factor of household i.

Assumption 2 $\beta_1 > \beta_2$.

The first-order conditions for a solution to the households' problem are:

$$c_{git}p_t = (1-\alpha)P_{it}C_{it} \tag{7}$$

$$(c_{pit} - c_0)(1+\tau) = \alpha P_{it}C_{it} \tag{8}$$

$$\frac{C_{it}^{-\sigma}}{P_{it}} E_t^{\mu} p_t \geq \beta_i \frac{C_{it+1}^{-\sigma}}{P_{it+1}} E_{t+1}^{\mu}(p_{t+1}(1-\delta) + r_{t+1}) \tag{9}$$

with $P_{it} = \frac{(1+\tau)^{\alpha} p_t^{1-\alpha}}{(1-\alpha)^{1-\alpha}\alpha^{\alpha}} + \frac{(1+\tau)c_0}{C_{it}}$ and (9) holding at equality when $a_{it} > 0$.

Equations (7) and (8) are standard optimality conditions for Cobb-Douglas utility functions: agents spend a constant share of their revenue in each good, augmented by the susbsitence consumption level for the polluting commodity. Equation (9) is the standard Euler equation.

Raising (8) at the power α and (7) at the power $(1-\alpha)$ allows to rewrite the budget constraint of household i:

$$P_{it}C_{it} + p_t(a_{it+1} - (1-\delta)a_{it}) = \frac{w_t}{2} + r_t a_{it} + T_{it} \tag{10}$$

3 Equilibrium

We focus on an equilibrium around the steady-state, meaning that $a_{1t} = K_t > 0 = a_{2t}$[1].

Using equations (2) and (3), we obtain:

$$K_t = \frac{K_{pt}}{N_{pt}} = \frac{K_{gt}}{N_{gt}} \tag{11}$$

$$p_t \equiv p = \frac{A_p}{A_g} \tag{12}$$

$$P_{it} = \frac{(1+\tau)^{\alpha} p_t^{1-\alpha}}{(1-\alpha)^{1-\alpha}\alpha^{\alpha}} + \frac{(1+\tau)c_0}{C_{it}} \tag{13}$$

Plugging (11) and (12) in (2) and (3) yields:

$$r_t = \eta A_p K_t^{\eta-1} \tag{14}$$

$$w_t = (1-\eta)A_p K_t^{\eta} \tag{15}$$

Using equations (10) and (9) in which we plug (14), (15) and the market clearing condition on the financial market for each household gives:

$$\frac{C_{1t}^{-\sigma}}{P_{1t}} E_t^{\mu} = \beta_1 \frac{C_{1t+1}^{-\sigma}}{P_{1t+1}} E_{t+1}^{\mu} \left(1 - \delta + \frac{r_{t+1}}{p}\right) \tag{16}$$

$$P_{1t}C_{1t} = \left(\frac{(1+\eta)}{2} + \frac{\alpha\varepsilon\tau}{1+\tau(1-\alpha)}\right) A_p K_t^{\eta} - (1 + \frac{\alpha_{\varepsilon}\tau}{1+\tau(1-\alpha)})p(K_{t+1} - (1-\delta)K_t) + \frac{2\varepsilon\tau(1+\tau)}{1+\tau(1-\alpha)}c_0 \tag{17}$$

$$\frac{C_{2t}^{-\sigma}}{P_{2t}} E_t^{\mu} > \beta_2 \frac{C_{2t+1}^{-\sigma}}{P_{2t+1}} E_{t+1}^{\mu} \left(1 - \delta + \frac{r_{t+1}}{p}\right) \tag{18}$$

$$P_{2t}C_{2t} = \frac{(1-\eta)(1+\tau)}{2(1+\tau(1-\alpha(1-\varepsilon)))} A_p K_t^{\eta} + (1-\varepsilon)\tau \left(\frac{\alpha P_{1t}C_{1t} + 2c_0(1+\tau)}{(1+\tau(1-\alpha(1-\varepsilon))}\right) \tag{19}$$

In order for C_1 and C_2 to be positive, we make the following assumption:

Assumption 3 *c_0 is bounded above by c_{01} whenever $\frac{1}{2\tau} + \frac{1-\alpha}{2} \geq \varepsilon$ and by c_{02} whenever $\varepsilon \geq \frac{1+\alpha}{2} - \frac{1}{2\tau}$*[2].

Households' income partly depends on the redistribution rate ε. When very few of the tax revenu is given back towards one household, the budget constraint gets tighter. In this cas, a too high

[1] proof can be found in the next section.

[2] As we focus on an equilibrium around the steady-state, we use equations (17) and (19) evaluated at the steady-state and look for the conditions under which we have $C_{1t} > 0$ and $C_{2t} > 0$.

subsistence level of consumption for the polluting good would make the constraint impossible to hold: the minimal level of consumption would be too high compared with the income. Hence an upper bound on c_0.

Equations (7), (8), (17), (19) and market clearing conditions allow to recover aggregate consumption:

$$c_{pt} = \sum_i c_{pit} = \frac{\alpha}{1+\tau(1-\alpha)}\left(A_p K_t^\eta - p(K_{t+1}-(1-\delta)K_t)\right) + \frac{2(1+\tau)}{1+\tau(1-\alpha)}c_0 \tag{20}$$

$$c_{gt} = \sum_i c_{git} = \frac{(1-\alpha)(1+\tau)}{p(1+\tau(1-\alpha))}\left(A_p K_t^\eta - p(K_{t+1}-(1-\delta)K_t) + 2\tau c_0\right) \tag{21}$$

Finally, equation (4) and the market clearing condition on the polluting sector yield $E_t = \frac{1}{\gamma c_{pt}}$.

Substituting (14) and (3) in equations (16) and (17) give the dynamic equations of the model:

$$K_{t+1} = \frac{\left(\frac{1+\eta}{2} + \frac{\alpha\varepsilon\tau}{(1+\tau(1-\alpha))}\right)A_p K_t^\eta + \frac{2\varepsilon\tau(1+\tau)}{1+\tau(1-\alpha)}c_0 - P_{1t}C_{1t}}{p\left(1+\frac{\alpha\varepsilon\tau}{(1+\tau(1-\alpha))}\right)} + (1-\delta)K_t \tag{22}$$

$$C_{1t+1}^\sigma = \beta_1 \frac{C_{1t}^\sigma P_{1t}}{P_{1t+1}}\left(\frac{\alpha P_{1t}C_{1t} + \frac{\alpha(1-\eta)}{2}A_p K_t^\eta + 2(1+\tau)c_0}{\alpha P_{1t+1}C_{1t+1} + \frac{\alpha(1-\eta)}{2}A_p K_{t+1}^\eta + 2(1+\tau)c_0}\right)^\mu \left(1-\delta+\eta A_g K_{t+1}^{\eta-1}\right) \tag{23}$$

which gives C_{1t+1} and K_{t+1} as functions of K_t and C_{1t}, and with P_{1t} a function of C_{1t}.

Definition 1 *Under assumptions 1 and 3, an equilibrium of the economy is a vector of prices* $\{\{p, P_{it}, \tau, w_t, r_t\}_{i=1,2}\}_{t=0}^\infty$ *and quantities* $\{\{c_{pit}, c_{git}, C_{it}, a_{it}, K_{pt}, K_{gt}, N_{pt}, N_{gt}, Y_{pt}, Y_{gt}\}_{i=1,2}\}_{t=0}^\infty$ *such that equations (22) and (23) are satisfied, optimality conditions, market-clearing conditions and the government's budget constraint hold. In this equilibrium, all variables are given by* K_t, C_{1t} *and parameters of the model.*

4 Steady State

In this section, we show the existence of a unique steady-state in which the most patient household holds a positive amount of capital, while the impatient one does not save at all. All steady-state variable are denoted with a star.

Taking the Euler equation for agent i, we can write $1 \geq \beta_i(1-\delta+\frac{r}{p})$, holding with equality if $a_i > 0$. Recall that $\beta_1 > \beta_2$, so that $\beta_1(1-\delta+\frac{r}{p}) > \beta_2(1-\delta+\frac{r}{p})$. Thus, we have $a_1^* = K^* > 0 = a_2^*$ as we assumed in the decentralized equilibrium. This a standard result in the literature (Becker (1980), Becker & Foias (1987), Sorger (1994)). Rewriting the left hand-side of the inequality: $\frac{r^*}{p^*} = \frac{1-\beta_1}{\beta_1} + \delta$.

As we found that p is a constant, we have $p^* = \frac{A_p}{A_g}$ and $P_i^* = \frac{(1+\tau)^\alpha p^{*(1-\alpha)}}{\alpha^\alpha(1-\alpha)^{1-\alpha}} + (1+\tau)\frac{c_0}{C_i^*}$.

Using (15) and (14), capital-labor ratios remain constant, with $K^* = \frac{K_p^*}{N_p^*} = \frac{K_g^*}{N_g^*} = \left(\frac{\eta A_g}{\frac{1}{\beta_1}-(1-\delta)}\right)^{\frac{1}{1-\eta}}$, and the wage rate can write $w^* = (1-\eta)A_p\left(\frac{\eta A_g}{\frac{1}{\beta_1}-(1-\delta)}\right)^{\frac{\eta}{1-\eta}}$.

Writing (7) and (8) at the steady state gives individual consumptions. Equations (17) and (19)-(21) allow to recover overall consumption for each household, as well as aggregate consumption in both sectors:

$$P_1^*C_1^* = K^*p^*\left(\frac{r^*/p^*}{\eta}\left(\frac{1+\eta}{2}+\frac{\alpha\varepsilon\tau}{1+\tau(1-\alpha)}\right)-\delta\left(1+\frac{\alpha\varepsilon\tau}{1+\tau(1-\alpha)}\right)\right)+\frac{2\varepsilon\tau(1+\tau)}{1+\tau(1-\alpha)}c_0 \quad (24)$$

$$P_2^*C_2^* = K^*p^*\left[\left(\frac{r^*/p^*}{\eta}-\delta\right)\frac{\alpha(1-\varepsilon)\tau}{1+\tau(1-\alpha)}+\frac{r^*/p^*}{\eta}\frac{1-\eta}{2}\right]+\frac{2(1-\varepsilon)\tau(1+\tau)}{1+\tau(1-\alpha)}c_0 \quad (25)$$

$$c_p^* = \frac{\alpha}{1+\tau(1-\alpha)}K^*p^*\left(\frac{r^*/p^*}{\eta}-\delta\right)+\frac{2(1+\tau)}{1+\tau(1-\alpha)}c_0 \quad (26)$$

$$c_g^* = \frac{(1-\alpha)(1+\tau)}{(1+\tau(1-\alpha))}K^*\left(\frac{r^*/p^*}{\eta}-\delta\right)+\frac{2\tau(1+\tau)(1-\alpha)}{p^*(1+\tau(1-\alpha))}c_0 \quad (27)$$

Finally, market clearing conditions give $Y_p^* = c_p^*$ and $Y_g^* = c_g^* + \delta K^*$, and pollution writes $E = \frac{1}{\gamma c_p^*}$.

Proposition 1 *Under assumptions 1 and 3, there exists a unique steady state in the economy* (K^*, C_1^*) *solution to (22)-(23) with constant prices and capital, and characterised by equations (24)-(27). At the steady-state, agent 2 is constrained and household 1 holds all the capital.*

5 Public Policy

The government has two tools to play on environmental quality and income and consumption inequalities: lump-sum transfers and commodity taxation. This part aims at analyzing the impact of both tools separately and analyze their impact on both pollution and inequalities.

5.1 Redistribution

Impact consumption and emissions

The effect of redistribution is straightforward. From equations (26) and (27), we see that redistribution has no impact on aggregate variables. Moreover, capital is only a function of parameters, so that redistribution has no impact on it either.

The only impact of ε is hence on individual variables, i.e on individual consumption. As consumption of both goods are functions of overall consumption, the only thing we have to look at is the impact of a change in redistribution on C_1^* and C_2^* respectively. From (24) and (25), giving more of the tax revenue to the worker has a positive effect on her consumption, while it is negative for the capitalist.

The redistribution therefore only impacts individual consumption through a pure redistribution effect: increasing redistribution towards the capitalist raises her disposable income, and hence her consumption in both goods proportionally, while it reduces the worker's consumption. Yet, both trends perfectly compensate each other so that aggregate consumption, and more broadly all aggregate variables, do not change, environmental damages included. Hence, it is possible for the government to reduce income and consumption inequalities by increasing redistribution towards the worker, while not harming environmental quality.

The absence of impact on pollution can be explained by one feature of our model: identical preferences. Indeed, Rausch & Schwartz (2016) show that when preferences are homothetic and identical, aggregate behaviors are similar to a single-agent behavior. Thus, changing the redistribution pattern has no impact on aggregate consumption as it is similar to giving back everything to a single agent. Allowing for heterogeneous preferences should allow for an aggregate impact of redistribution. We set the following assumption:

Assumption 4 *Households now differ in their preferences through their spending share on the polluting commodity. More specifically, we assume that the worker spends a higher share of her income on the polluting commodity, i.e* $\alpha_2 > \alpha_1$.

Proposition 2 *Under assumptions Assumption 1 to 4, a decrease in* ε *decreases clean consumption and increases polluting consumption. Reducing income inequalities through redistribution therefore harms environmental quality.*

As for the homogeneous preferences case, changes in consumption occur only through a pure revenue effect. As we assume $\alpha_2 > \alpha_1$, the increase in polluting consumption for the worker is higher than the decrease for the capitalist, while the reverse occurs for clean commodity consumption. Therefore, c_p^* increases and c_g^* decreases. As environmental quality directly depends on clean commodity consumption, it declines after a switch in ε.

Impact on welfare

Changing redistribution affects individual welfare through consumption and environmental quality. More precisely:

$$\frac{dU_i^*}{d\varepsilon} = \mu \frac{dE^*}{d\varepsilon} E^{*\mu-1} \frac{C_i^{*1-\sigma}}{1-\sigma} + \frac{dC_i}{d\varepsilon} C_i^{*-\sigma} E^{*\mu} \tag{28}$$

Proposition 3 *Under assumptions 1 to 4, decreasing income inequalities through redistribution:*

- *decreases welfare for the capitalist and increases it for the worker when* $\mu < \mu^*$,
- *decreases welfare for both households when* $\mu \geq \mu^*$.

with $\mu^* = (1-\sigma)\frac{\partial C_2^*/\partial\varepsilon}{\partial c_p^*/\partial\varepsilon}\frac{c_p^*}{C_2^*}$.

From Proposition 2, giving more of the tax revenue to the worker harms environmental quality, which has a negative impact on households' utilities. As the capitalist decreases her overall consumption, her utility always shrinks. Yet, the worker increases her consumption in both commodities: the impact on her utility hence depends on how much does environmental quality matter in the utility function, i.e on the size of μ. When $\mu < \mu^*$, the impact on consumption overrules the impact on environmental quality, so that U_2^* increases. The reverse occurs when environmental quality matters a lot ($\mu > \mu^*$): the negative impact on E^* prevails.

5.2 Taxation

There are two effects at play when increasing the tax rate in our model: a price effect and a redistribution effect. By increasing the consumer price for the polluting commodity, purchasing power of both households is lowered so that for the same income, they consume less. Increasing the consumer price also leads to a substitution between the two goods: agents increase their consumption in the clean good as its relative price decreases. Increasing the tax rate leads to more government revenue, and hence to higher transfers to households. Everything else equal, this rise in income allows for a higher level of consumption: this is the redistribution effect.

Impact on consumption and environmental quality

We first investigate the effect of an increase in τ on aggregate consumptions, c_p^* and c_g^*. Denoting $\overline{c_0} = \frac{K^* p^* (\frac{r^*/p^*}{\eta} - \delta\eta)(1-\alpha)}{2\eta}$,

Proposition 4 *Under assumptions 1 and 3, an increase in commodity taxation always leads to an increase in clean consumption. Polluting consumption decreases when $c_0 < \overline{c_0}$ and increases for $c_0 > \overline{c_0}$.*

The increase in c_g^* comes from the substitution effect: an increase in taxation makes the clean good relatively cheaper than the polluting one, so that households substitute one consumption to another. The mixed effect on polluting consumption comes from the price and redistribution effects. The former is negative, while the latter is positive. A low subsistence level pushes the redistribution effect down so that the price effect dominates. The reverse occurs when subsistence consumption is high.

Let us now focus on individual behaviors. Using equation (7) at the steady-state, we have:

$$\frac{dc_{gi}^*}{d\tau} = \frac{(1-\alpha)}{p^*} \frac{d(P_i^* C_i^*)}{d\tau} > 0$$

for any household i. Clean consumption does not always increase only at the aggregate level, but also at the individual one, which corresponds again to the susbstitution effect of the tax.

Results for individual polluting consumption are summed up in the following proposition, with $\varepsilon^* = \frac{1}{2}\left(1 + \eta \frac{\frac{1-\beta_1}{\beta_1}}{\frac{1-\beta_1}{\beta_1} + (1-\eta)\delta}\right)$:

Proposition 5 *Under assumptions 1 and 3 and for*

- $0 \leq c_0 < \overline{c_0}$:
    - *When* $\varepsilon < \varepsilon^*$, c_{p1}^* *decreases and* c_{p2}^* *decreases for* $\alpha < \underline{\alpha_2}$ *and increases for* $\alpha > \overline{\alpha_2}$;
    - *When* $\varepsilon > \varepsilon^*$, c_{p2}^* *decreases and* c_{p1}^* *decreases for* $\alpha < \underline{\alpha_1}$ *and increases for* $\alpha > \overline{\alpha_1}$

- $c_0 > \overline{c_0}$:
    - *When* $\varepsilon < \varepsilon^*$, c_{p2}^* *increases and* c_{p1}^* *decreases for* $\alpha < \underline{\alpha_1}$ *and increases for* $\alpha > \overline{\alpha_1}$;
    - *When* $\varepsilon > \varepsilon^*$, c_{p1}^* *increases and* c_{p2}^* *decreases for* $\alpha < \underline{\alpha_2}$ *and increases for* $\alpha > \overline{\alpha_2}$.

From Proposition 5, three variables are key when analyzing which effect dominates the other: the redistribution rate, the size of subsistence consumption and the spending share on the polluting commodity. These three variables play a role on the redistribution effect, as the tax revenue depends on both c_0 and α.

When the government redistributes few of the tax revenue towards household i and that c_0 is low, no matter α the size of the redistribution effect is very low, so that the price effect dominates and polluting consumption decreases. On the other hand, when a high share of the tax revenue is redistributed and c_0 is high, the size of the redistribution effect becomes so big that it overrules the price effect, leading to an increase in polluting consumption for that household.

When c_0 is low (resp. high) and a high (resp. low) share of the tax revenue is redistributed towards household i, then which effect dominates the other depends on the size of α. When α is low, the redistribution effect is pushed down by c_0 and α, so that the price effect dominates. However, a high α pushes it up as a lot of the tax revenue is given to the household, hence the redistribution effect dominates.

impact on welfare

Changing the commodity tax rate affects individual welfare through consumption and environmental quality:

$$\frac{dU_i^*}{d\tau} = \mu \frac{dE^*}{d\tau} E^{*\mu-1} \frac{C_i^{*1-\sigma}}{1-\sigma} + \frac{dC_i}{d\tau} C_i^{*-\sigma} E^{*\mu} \tag{29}$$

Proposition 6 *Under assumptions 1, 3 and* **??**, *increasing the tax rate on the polluting commodity:*

- *For* $\mu < min(\mu_1^*(\tau, \varepsilon), \mu_2^*(\tau, \varepsilon))$:
    - *When* $c_0 < \overline{c_0}$: *welfare decreases for the capitalist when* $\varepsilon < \varepsilon_1$, *and decreases for the worker when* $\varepsilon > \frac{1+\alpha}{2}$;
    - *When* $c_0 > \overline{c_0}$: *welfare increases for both households whenever* $\varepsilon \in \left(\frac{1-\alpha}{2}, \varepsilon_1\right)$.

- *For* $\mu > \max(\mu_1^*(\tau, \varepsilon), \mu_2^*(\tau, \varepsilon))$:
    - *When* $c_0 < \overline{c_0}$ *and and* $\alpha > \hat{\alpha}$ *if* $\varepsilon \in \left(\varepsilon_1, \frac{1-\alpha}{2}\right)$, *welfare increases for both households;*
    - *When* $c_0 > \overline{c_0}$, *welfare decreases for both households.*

When subsistence consumption is low, the worker's welfare increases when she receives a high share of the tax revenue (both consumption and environmental quality increase), or when few of the tax revenue is redistributed but environmental quality matters a lot. The same mechanism applies for the capitalist: her welfare increases when a lot of the tax revenue is redistributed towards her, or when a lower share of the tax revenue is redistributed and she spends a high share of her income on the polluting commodity (a high α pushes the redistribution effect up so that consumption increases). When she receives a low share of the tax revenue and environmental quality matters a lot, the positive effect on environmental quality overrules the negative effect on consumption. For both households, welfare decreases when a low share of the tax revenue is redistributed towards them, and environmental quality does not matter a lot, so that the decrease in consumption offsets the increase in environmental quality.
When subsistence consumption is high, welfare decreases for both households whenever environmental quality matters a lot. When it does not, then it increases for both households under intermediate values of ε.

When environmental quality does not matters a lot and subsistence consumption is low, increasing the tax rate on the polluting commodity and redistributing a lot of the revenue towards the worker reconciles the tradeoff between environmental preservation and inequality reduction we found when playing on transfers. More generally, when subsistence consumption is low, there always exists a way to reconcile this tradeoff by redistributing more of the tax revenue to the worker[3]. On the other hand, a high level of subsistence consumption does not allow for this as environmental quality will always decrease. More than that, when environmental quality matters a lot, taxation is bad both for the environment and for welfare, even if income inequalities are reduced.

6 Local Stability

We now characterize the role of subsistence consumption and the environmental externality on stability properties near the steady-state of (22)-(23). For that, we look at the log-linearized system evaluated at the steady-state:

$$\begin{bmatrix} \tilde{K_{t+1}} \\ \tilde{C_{t+1}} \end{bmatrix} = \begin{bmatrix} \Omega_1 & -\Omega_2 \\ \Omega_3 & \Omega_4 \end{bmatrix} \begin{bmatrix} \tilde{K_t} \\ \tilde{C_t} \end{bmatrix}$$

[3] $\frac{dP_2^* C_2^*}{d\tau} > \frac{dP_1^* C_1^*}{d\tau}$ as long as $\varepsilon < \frac{1}{2}$.

with

$$\Omega_1 = \frac{\frac{1+\eta}{2} + \frac{\alpha\varepsilon\tau}{1+\tau(1-\alpha)}}{p\left(1+\frac{\alpha\varepsilon\tau}{1+\tau(1-\alpha)}\right)}\left(\frac{1}{\beta_1} - 1 + \delta\right) + (1-\delta) \tag{30}$$

$$\Omega_2 = \frac{(1+\tau)^{\alpha} p^{1-\alpha} C_1^*}{\alpha^{\alpha}(1-\alpha)^{1-\alpha} p\left(1+\frac{\alpha\varepsilon\tau}{1+\tau(1-\alpha)}\right) K^*} \tag{31}$$

$$\Omega_3 = \frac{\mu\alpha\eta\frac{(1-\eta)}{2} A_p K^{*\eta} C_1^{*-\sigma}\left(\left(\frac{1+\eta}{2} + \frac{\alpha\varepsilon\tau}{1+\tau(1-\alpha)}\right)\left(\frac{1}{\beta_1} - 1 + \delta\right) - \delta\right) - \frac{\beta_1}{p}\kappa C_1^{*-\sigma} K^* f''(K^*)}{(1-\tau)c_0 C_1^{*-\sigma-1} - \sigma C_1^{*-\sigma} - \mu\alpha C_1^{*1-\sigma}\frac{(1+\tau)^{\alpha}p^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}} \tag{32}$$

$$\Omega_4 = 1 + \frac{\frac{\beta_1}{p} C_1^{*1-\sigma} K^* f''(K^*)(1+\tau)^{\alpha} p^{1-\alpha} - \mu\alpha\frac{(1-\eta)}{2}(1+\tau)^{\alpha}p^{1-\alpha}C_1^{*1-\sigma}\left(\frac{1}{\beta_1} - 1 + \delta\right)}{\alpha^{\alpha}(1-\alpha)^{1-\alpha} p\left(1+\frac{\alpha\varepsilon\tau}{1+\tau(1-\alpha)}\right)\left((1-\tau)c_0 C_1^{*-\sigma-1} - \sigma C_1^{*-\sigma} - \mu\alpha C_1^{*1-\sigma}\frac{(1+\tau)^{\alpha}p^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}\right)} \tag{33}$$

We now explore the role of the externality μ and subsistence consumption c_0 on the stability properties of the economy. Following Grandmont et al. (1998), we use the trace and the determinant to explore the stability properties of the system, given by:

$$Tr = \Omega_1 + \Omega_4 \tag{34}$$

$$D = \Omega_1 - \frac{\mu\alpha\frac{(1-\eta)}{2}(1+\tau)^{\alpha}p^{1-\alpha}C_1^{*1-\sigma}\left(\frac{1}{\beta_1} - 1 + \delta\right)}{\alpha^{\alpha}(1-\alpha)^{1-\alpha} p\left(1+\frac{\alpha\varepsilon\tau}{1+\tau(1-\alpha)}\right)\left((1-\tau)c_0 C_1^{*-\sigma-1} - \sigma C_1^{*-\sigma} - \mu\alpha C_1^{*1-\sigma}\frac{(1+\tau)^{\alpha}p^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}\right)} \tag{35}$$

To do so, we use the characteristic polynomial $P(x) = x^2 - xTr + D$ evaluated at -1, 0 and 1.

GRAPH HERE.

6.1 Local dynamics when $\mu > 0$ and $c_0 = 0$

When environmental quality enters households' utilities but there is no subsistence consumption for the polluting good, the trace and the determinant are given by:

$$Tr = 1 + \Omega_1 + \frac{\frac{\beta_1}{p} C_1^{*1-\sigma} K^* f''(K^*)(1+\tau)^{\alpha} p^{1-\alpha} - \mu\alpha\frac{(1-\eta)}{2}(1+\tau)^{\alpha}p^{1-\alpha}C_1^{*1-\sigma}\left(\frac{1}{\beta_1} - 1 + \delta\right)}{\alpha^{\alpha}(1-\alpha)^{1-\alpha} p\left(1+\frac{\alpha\varepsilon\tau}{1+\tau(1-\alpha)}\right)\left(-\sigma C_1^{*-\sigma} - \mu\alpha C_1^{*1-\sigma}\frac{(1+\tau)^{\alpha}p^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}\right)} \tag{36}$$

$$D = \Omega_1 + \frac{\mu\alpha\frac{(1-\eta)}{2}(1+\tau)^{\alpha}p^{1-\alpha}C_1^{*1-\sigma}\left(\frac{1}{\beta_1} - 1 + \delta\right)}{\alpha^{\alpha}(1-\alpha)^{1-\alpha} p\left(1+\frac{\alpha\varepsilon\tau}{1+\tau(1-\alpha)}\right)\left(\sigma C_1^{*-\sigma} + \mu\alpha C_1^{*1-\sigma}\frac{(1+\tau)^{\alpha}p^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}\right)} \tag{37}$$

This gives the following stability properties:

Proposition 7 *Under Assumptions 1 and 3, a steady-state with no subsistence consumption and taking into account the externality is a saddle.*

Accounting for the externality with homothetic preferences ensures saddle-path stability of the system. In this case, taxation and redistribution have no role to play in the stability properties of the economy.

6.2 Local dynamics when $c_0 > 0$ and $\mu = 0$

When the externality is not taken into account, the trace and the determinant simplify to:

$$Tr = 1 + \Omega_1 - \frac{\beta_1 f''(K^*) C_1^* (1+\tau)^\alpha p^{1-\alpha}}{\alpha^\alpha (1-\alpha)^{1-\alpha} \left(1 + \frac{\alpha\varepsilon\tau}{1+\tau(1-\alpha)}\right)(\sigma C_1^* - (1+\tau)c_0)} \quad (38)$$

$$D = \Omega_1 \quad (39)$$

From that, we deduce the following stability properties:

Proposition 8 *Under Assumptions 1 and 3, a steady-state under no environmental externality and a positive level of subsistence consumption has the following stability properties:*

- *when* $c_0 < c_0^*(\tau, \varepsilon)$*, the steady-state is a saddle;*
- *when* $c_0^*(\tau, \varepsilon) < c_0 < c_0^{**}(\tau, \varepsilon)$*, the steady-state is a saddle;*
- *when* $c_0 > c_0^{**}(\tau, \varepsilon)$*, the steady-state is a source for* $p < \overline{p}$ *and a sink otherwise.*

A flip bifurcation occurs at $c_0 = c_0^{**}$ *and* $p > \overline{p}$*, and a Hopf birfurcation arises when* $c_0 > c_0^{**}$ *and* $p = \overline{p}$*.*

When accounting for subsistence consumption but not for the externality, some instability can be brought depending on the level of c_0 and p. As long as $c_0 < c_0^{**}$, saddle-path stability is ensured. When $c_0 < c_0^*$, the Trace value lies on the RHS of the graph, while on the LHS when $c_0^*(\tau, \varepsilon) < c_0 < c_0^{**}(\tau, \varepsilon)$. When $c_0 > c_0^{**}(\tau, \varepsilon)$, local unstability arises when $p < \overline{p}$, and indeterminacy occurs whenever $p > p^*$. As both c_0^{**} and $\overline{p}$ depend on τ and ε, there is a role to play for both taxation and rdistirbution in the maintaining of stability and determinacy of the equilibrium.

6.3 Local dynamics when $c_0 > 0$ and $\mu > 0$

In this case, the trace and the determinant are given by (34)-(35). The economy has the following stability properties:

Proposition 9 *Under Assumptions 1 and 3, a steady-state with positive subsistence consumption and externality is a source when* $c_0 > c_0^*$ *and* $\mu < \mu^*$*, and is a saddle otherwise.*

Accounting for both the environmental externality and subsistence consumption brings back determinacy of the equilibrium, but can still lead to unstability. Again, unstability arises when susbistence consumption is high and pollution does not matter a lot in the utility function. Therefore, the tax and redistribution rates again have a role to play in the stability of the system.

7 Concluding Remarks

In this paper, we investigate the impact of environmental commodity taxation and redistribution on pollution and income inequality. For that, we build a two-sector Ramsey model with heterogeneous households, an environmental externality and a subsistence level of consumption for the polluting good.

After characterizing the intertemporal equilibrium, we show that there exists a unique steady-state in which the most patient household holds all the capital. We then discuss the impact of taxation and redistribution on pollution and inequality. Considering the worker spends a higher share of her income on the polluting commodity, increasing the share of the tax revenue she receives increases pollution: there is a trade-off between inequality and pollution reduction. Focusing on taxation, we find that increasing the tax rate does not reduce pollution when the level of susbistence consumption is too high.

Analyzing the local dynamics, we show that the level of subsistence consumption can lead to indeterminacy. This parameter, coupled with the externality parameter, are also source of un-stability when theyr are both taken into account. In this case, the environmental fiscal policy has a role to play to maintain the stability of the equilibrium. Hence, policy makers must carefully handle taxation and redistribution to avoid instability, and they also must take into account subsistence consumption when implementing environmental tax reforms.

Appendix

Household's Problem

The household's problem writes:

$$\begin{aligned} \max_{a_{it+1}, c_{pit}, c_{git}} \quad & \sum_{t=0}^{\infty} \beta_i^t U_i(c_{pit}, c_{git}, E_t) \\ \text{subject to} \quad & c_{pit}(1+\tau) + c_{git}p_t + p_t(a_{it+1} - (1-\delta)a_{it}) = w_t n_i + r_t a_{it} + T_{it} \\ & a_{it} \geq 0 \end{aligned}$$

Using the following Lagrangian to solve the optimizaton problem:

$$\mathbb{L} = E_t^{\mu} \frac{(c_{pit}^{\alpha_i} c_{git}^{1-\alpha_i})^{1-\sigma}}{1-\sigma} + \lambda_{it}^{BC}(w_t n_i + r_t a_{it} + T_{it} - c_{pit}(1+\tau) - c_{git}p_t - p_t(a_{it+1} - (1-\delta)a_{it})) + \lambda_{it}^{FC} a_{it}$$

Optimality conditions:

$$\begin{aligned} \frac{1-\alpha}{c_{git}} C_{it}^{1-\sigma} E_t^{\mu} &= \lambda_{it}^{BC} p_t \\ \frac{\alpha}{c_{pit}} C_{it}^{1-\sigma} E_t^{\mu} &= \lambda_{it}^{BC}(1+\tau) \\ \lambda_{it}^{B} C p_t &= \beta_i \lambda_{it+1}^{BC}(p_{t+1}(1-\delta) + r_{t+1}) + \beta_i \lambda_{it+1}^{FC} \end{aligned}$$

Taking the first FOC at the power α and th second FOC at the power $1-\alpha$, then multiplicating them yields for agent i:

$$E_t^{\mu} C_{it}^{1-\sigma}(1-\alpha_i)^{1-\alpha_i} \alpha_i^{\alpha_i} = \lambda_{it}^{BC}(1+\tau)^{\alpha_i} p_t^{1-\alpha} c_{pit}^{\alpha_i} c_{git}^{1-\alpha_i}$$

Recall that $C_{it} = c_{pit}^{\alpha_i} c_{git}^{1-\alpha_i}$, so that the equation can be rewritten:

$$E_t^{\mu} C_{it}^{-\sigma} \frac{(1-\alpha_i)^{1-\alpha_i} \alpha_i^{\alpha_i}}{(1+\tau)^{\alpha_i} p_t^{1-\alpha_i}} = \lambda_{it}^{BC}$$

As $P_{it} = \frac{(1+\tau)^{\alpha_i} p_t^{1-\alpha_i}}{(1-\alpha_i)^{1-\alpha_i} \alpha_i^{\alpha_i}}$, we obtain $\lambda_{it}^{BC} = \frac{C_{it}^{-\sigma}}{P_t} E_t^{\mu}$.

Dividing the two first FOCs and rearranging gives $c_{git} p_t = \frac{1-\alpha_i}{\alpha_i} c_{pit}(1+\tau)$.
Denote $I_{it} = w_t n_i + r_t a_{it} - (a_{it+1} - (1-\delta) a_{it})$. Plugging the previous equation for $c_{git} p_t$ in the budget constraint of agent i yields:

$$c_{pit} = \frac{\alpha_i}{1+\tau} I_{it}$$
$$c_{git} = \frac{1-\alpha_i}{p_t} I_{it}$$

The ex post budget constraint can be written $P_{it} C_{it} = I_{it}$.
Recall $C_{it} = c_{pit}^{\alpha_i} c_{git}^{1-\alpha_i}$, so that $C_{it} = \left(\frac{\alpha_i}{1+\tau} I_{it}\right)^{\alpha_i} \left(\frac{1-\alpha_i}{p_t} I_{it}\right)^{1-\alpha_i}$.
Plugging this into the ex post budget constraint and rearranging yields $P_{it} = \frac{(1+\tau)^{\alpha_i} p_t^{1-\alpha_i}}{(1-\alpha_i)^{1-\alpha_i} \alpha_i^{\alpha_i}}$.

Proof of Proposition 2

Under heterogeneous preferences, steady-state variables become:

$$P_1^* C_1^* = K^* p^* \left(\frac{r^*/p^*}{\eta} \left(\frac{(1+\eta)(1+\tau(1-\alpha_2))+2\varepsilon\alpha_2\tau}{2(1+\tau(1-\alpha_2(1-\varepsilon)-\alpha_1\varepsilon))}\right) - \delta\left(\frac{1+\tau(1-\alpha_2(1-\varepsilon))}{1+\tau(1-\alpha_2(1-\varepsilon)-\alpha_1\varepsilon)}\right)\right) + \frac{2\varepsilon\tau(1+\tau)}{1+\tau(1-\alpha_2(1-\varepsilon)-\alpha_1\varepsilon)} c_0 \quad (40)$$

$$P_2^* C_2^* = K^* p^* \left(\frac{r^*/p^*}{\eta} \left(\frac{(1+\tau(1+\alpha_1(1-2\varepsilon))-\eta(1+\tau(1-\alpha_1))}{2(1+\tau(1-\alpha_2(1-\varepsilon)-\alpha_1\varepsilon))}\right) - \delta\left(\frac{\alpha_1(1-\varepsilon)\tau}{1+\tau(1-\alpha_2(1-\varepsilon)-\alpha_1\varepsilon)}\right)\right) + \frac{2(1-\varepsilon)\tau(1+\tau)}{1+\tau(1-\alpha_2(1-\varepsilon)-\alpha_1\varepsilon)} c_0 \quad (41)$$

$$c_p^* = K^* p^* \left(\frac{r^*/p^*}{\eta} \left(\frac{\alpha_1+\alpha_2-\eta(\alpha_2-\alpha_1)}{2(1+\tau(1-\alpha_2(1-\varepsilon)-\alpha_1\varepsilon))}\right) - \delta\left(\frac{\alpha_1}{1+\tau(1-\alpha_2(1-\varepsilon)-\alpha_1\varepsilon)}\right)\right) + \frac{2(1+\tau)}{1+\tau(1-\alpha_2(1-\varepsilon)-\alpha_1\varepsilon)} c_0 \quad (42)$$

$$c_g^* = K^* \left(\frac{r^*/p^*}{\eta} \left(1 - \frac{(\alpha_1+\alpha_2)}{2(1+\tau(1-\alpha_2(1-\varepsilon)-\alpha_1\varepsilon))}\right) - \delta\left(1 - \frac{\alpha_1}{1+\tau(1-\alpha_2(1-\varepsilon)-\alpha_1\varepsilon)}\right)\right) + \frac{2\tau(1+\tau)((1-\alpha_2(1-\varepsilon)-\alpha_1\varepsilon)}{p^*(1+\tau(1-\alpha_2(1-\varepsilon)-\alpha_1\varepsilon))} c_0 \quad (43)$$

From equations (26) and (27), we get:

$$\frac{dc_p^*}{d\varepsilon} = \frac{-(\alpha_2-\alpha_1)\tau}{(1+\tau(1-\alpha_2(1-\varepsilon)-\alpha_1\varepsilon))^2} \left[K^* p^* \left(\frac{r^*/p^*}{\eta} \frac{(\alpha_1+\alpha_2-\eta(\alpha_2-\alpha_1))}{2} - \delta\alpha_1\right) + 2(1+\tau)c_0\right] \quad (44)$$

$$\frac{dc_g^*}{d\varepsilon} = \frac{(\alpha_2-\alpha_1)\tau}{(1+\tau(1-\alpha_2(1-\varepsilon)-\alpha_1\varepsilon))^2} \left[K^* p^* \left(\frac{r^*/p^*}{\eta} \frac{(\alpha_1+\alpha_2)}{2} - \delta\alpha_1\right) + 2(1+\tau)c_0\right] \quad (45)$$

As we assumed $\alpha_2 > \alpha_1$, we have $\frac{dc_p^*}{d\varepsilon} > 0$ and $\frac{dc_g^*}{d\varepsilon} < 0$ when decreasing ε.

Proof of Proposition 3

Taking the derivatives of C_1^* and C_2^* with respect to ε yields:

$$\frac{dC_1^*}{d\varepsilon} = \frac{\alpha_1^{\alpha_1}(1-\alpha_1)^{(1-\alpha_1)}}{(1+\tau)^{\alpha_1} p^{*(1-\alpha_1)}} \left(K^* p^* \left(\frac{r^*/p^*}{\eta} \left(\frac{\tau(1+\tau(1-\alpha_2))(\alpha_2(1-\eta)+\alpha_1(1+\eta))}{2(1+\tau(1-\alpha_1(1)\varepsilon)-\alpha_1\varepsilon))^2}\right) - \delta\left(\frac{\alpha_1\tau(1+\tau(1-\alpha_2))}{(1+\tau(1-\alpha_1(1)\varepsilon)-\alpha_1\varepsilon))^2}\right)\right) + \frac{2\tau(1+\tau(1-\alpha_2)}{(1+\tau(1-\alpha_1(1)\varepsilon)-\alpha_1\varepsilon))^2} c_0\right)$$

$$\frac{dC_2^*}{d\varepsilon} = \frac{\alpha_2^{\alpha_2}(1-\alpha_2)^{(1-\alpha_2)}}{(1+\tau)^{\alpha_2} p^{*(1-\alpha_2)}} \left(K^* p^* \left(\frac{r^*/p^*}{\eta} \left(\frac{\tau(1+\tau(1-\alpha_1))(2\alpha_1-(\alpha_2-\alpha_1)(1-\eta))}{2(1+\tau(1-\alpha_2(1-\varepsilon)-\alpha_1\varepsilon))^2}\right) + \delta\left(\frac{\alpha_1\tau(1+\tau(1-\alpha_1))}{(1+\tau(1-\alpha_1(1)\varepsilon)-\alpha_1\varepsilon))^2}\right)\right) - \frac{2\tau(1+\tau(1-\alpha_1)}{(1+\tau(1-\alpha_1(1)\varepsilon)-\alpha_1\varepsilon))^2} c_0\right)$$

Following (42), $\frac{dE^*}{d\varepsilon}$ is positive. Using 40 and 41, we obtain $\frac{dC_1^*}{d\varepsilon} > 0$ and $\frac{dC_2^*}{d\varepsilon} < 0$. As $\frac{dU_i}{d\varepsilon}$ depends on $\frac{dE^*}{d\varepsilon}$ and $\frac{dC_i^*}{d\varepsilon}$, we have that U_1 always decreases. As we have 2 forces going in opposite directions for the worker, the signe of $\frac{dU_2^*}{d\varepsilon}$ will depend on the size of μ.

Taxation. From (26) and (27):

$$\frac{dc_p^*}{d\tau} = \frac{-\alpha(1-\alpha)}{(1+\tau(1-\alpha))^2} K^* p^* \left(\frac{r^*/p^*}{\eta} - \delta\right) + \frac{2\alpha}{(1+\tau(1-\alpha))^2} c_0$$

$$\frac{dc_g^*}{d\tau} = \frac{1-\alpha}{p(1+\tau(1-\alpha))^2} \left(K^* p^* \left(\frac{r^*/p^*}{\eta} - \delta\right) + 2(1+2\tau(1+\tau^c(1-\alpha)))c_0\right)$$

Proof of Proposition 5

For the capitalist

The derivative of c_{p1}^* with respect to τ is a second degree polynomial. Polluting consumption decreases for:

$$\tau^2\left(K^*p^*(1-\alpha)\left(\frac{r^*/p^*}{\eta} - \eta\delta\right) + \left(\frac{r^*/p^*}{\eta} + \eta\left(\frac{1}{\beta_1} - 1 - \delta\right)\right)\frac{(1-\alpha)}{2\alpha\varepsilon} - \frac{2\eta c_0}{\alpha}\right) + \tau\left(\left(\frac{r^*/p^*}{\eta} + \eta\left(\frac{1}{\beta_1} - 1 - \delta\right)\right)\frac{(1-\alpha)}{\alpha\varepsilon} - \frac{4\eta c_0}{\alpha}\right) - \left(K^*p^*\left(\frac{r^*/p^*}{\eta} - \eta\delta - \frac{\left(\frac{r^*/p^*}{\eta}+\eta\left(\frac{1}{\beta_1}-1-\delta\right)\right)}{2\alpha\varepsilon}\right) + \frac{2\eta c_0}{\alpha}\right) > 0,$$

positive when $\left(2\left(\frac{r^*/p^*}{\eta} - \eta\delta\right) - \frac{\left(\frac{r^*/p^*}{\eta}+\eta\left(\frac{1}{\beta_1}-1-\delta\right)\right)}{\varepsilon}\right)\left(K^*p^*(\frac{r^*/p^*}{\eta} - \delta\eta)(1-\alpha) - 2\eta c_0\right) > 0$.

When this is negative, the sign of the polynomial is equal to the sign of $K^*p^*(1-\alpha)\left(\frac{r^*/p^*}{\eta} - \eta\delta\right) + \left(\frac{r^*/p^*}{\eta} + \eta\left(\frac{1}{\beta_1} - 1 - \delta\right)\frac{(1-\alpha)}{2\alpha\varepsilon}\right) - \frac{2\eta c_0}{\alpha}$, which is positive for $0 < \alpha < \hat{\alpha_1}$ and negative for $\hat{\alpha_1} < \alpha < 1$.

- $\varepsilon < \varepsilon^*$:
  - When $c_0 < \overline{c_0}$, the discriminant is negative and $\alpha < \hat{\alpha}$, so the polynomial is positive, meaning c_{p1}^* decreases.
  - When $c_0 > \overline{c_0}$, the discriminant is positive and there are 2 roots:

$$\tau_1^1 = \frac{\frac{4\eta c_0}{\alpha} - \left(\frac{r^*/p^*}{\eta}+\eta\left(\frac{1}{\beta_1}-1-\delta\right)\right)\frac{(1-\alpha)}{\alpha\varepsilon} - \sqrt{2K^*p^*\left(2\left(\frac{r^*/p^*}{\eta}-\eta\delta\right)-\frac{\left(\frac{r^*/p^*}{\eta}+\eta\left(\frac{1}{\beta_1}-1-\delta\right)\right)}{\varepsilon}\right)\left(K^*p^*(\frac{r^*/p^*}{\eta}-\delta\eta)(1-\alpha)-2\eta c_0\right)}}{2\left(K^*p^*(1-\alpha)\left(\frac{r^*/p^*}{\eta}-\eta\delta+\left(\frac{r^*/p^*}{\eta}+\eta\left(\frac{1}{\beta_1}-1-\delta\right)\right)\frac{(1-\alpha)}{2\alpha\varepsilon}\right)-\frac{2\eta c_0}{\alpha}\right)}$$

$$\tau_2^1 = \frac{\frac{4\eta c_0}{\alpha} - \left(\frac{r^*/p^*}{\eta}+\eta\left(\frac{1}{\beta_1}-1-\delta\right)\right)\frac{(1-\alpha)}{\alpha\varepsilon} + \sqrt{2K^*p^*\left(2\left(\frac{r^*/p^*}{\eta}-\eta\delta\right)-\frac{\left(\frac{r^*/p^*}{\eta}+\eta\left(\frac{1}{\beta_1}-1-\delta\right)\right)}{\varepsilon}\right)\left(K^*p^*(\frac{r^*/p^*}{\eta}-\delta\eta)(1-\alpha)-2\eta c_0\right)}}{2\left(K^*p^*(1-\alpha)\left(\frac{r^*/p^*}{\eta}-\eta\delta+\left(\frac{r^*/p^*}{\eta}+\eta\left(\frac{1}{\beta_1}-1-\delta\right)\right)\frac{(1-\alpha)}{2\alpha\varepsilon}\right)-\frac{2\eta c_0}{\alpha}\right)}$$

  We know $a > 0$ for $0 < \alpha < \hat{\alpha_1}$ and $a < 0$ for $\hat{\alpha_1} < \alpha < 1$, and $\hat{\alpha_1} < 0$ for $\varepsilon > \frac{K^*p^*B}{4\eta c_0}$. When $\varepsilon > \frac{K^*p^*B}{4\eta c_0}$, $a < 0$, the numerators of τ_1^1 and τ_2^1 are always positive so $\tau_1^1 < 0$ and $\tau_2^1 < 0$ for any value of α and c_{p1}^* increases.

  When $\varepsilon < \frac{K^*p^*B}{4\eta c_0}$, the numerator of τ_1^1 is an increasing function of α, and evaluating the numerator of τ_1^1 at $\hat{\alpha_1}$, we find it is equal to zero. Hence $\tau_1^1 < 0$ for any value of α. the numerator of τ_2^1 has an inverted-U shape in α, and we know that it is negative at the minimum value of α, and positive when $\alpha = 1$. Also, when $\alpha \to \hat{\alpha}$, $\tau_2^1 \to +\infty$, so there exists an $\underline{\alpha_1} < \hat{\alpha}$ such that $\tau_2^1 > 0$ and a $\overline{\alpha_1} < \hat{\alpha}$ such that $\tau_2^1 > 1$. We can summarize:

α	c_{p1}^*
$0 < \alpha < \underline{\alpha_1}$	decreases
$\underline{\alpha_1} < \alpha < \overline{\alpha_1}$	has an inverted U-shape
$\overline{\alpha_1} < \alpha$	increases

- $\varepsilon > \varepsilon^*$:

  - When $c_0 < \overline{c_0}$, the discriminant is positive and there are 2 roots, τ_1^1 and τ_2^1.
    When $\varepsilon > \frac{K^*p^*B}{4\eta c_0}$, $a < 0$, and the numerator of τ_1^1 is a decreasing function of α. At the maximum value α can take, the numerator is positive, so thenumerator if always positive on the interval and $\tau_1^1 < 0$ always. The numerator of $\tau_2^1 > 0$ always, so $\tau_2^1 < 0$ and c_{p1}^* increases.
    When $\varepsilon < \frac{K^*p^*B}{4\eta c_0}$, the numerator of τ_1^1 has an inverted U shape in α, and is positive at $max(\alpha)$. We know then it crosses 0 once at $\hat{\alpha_1}$, so $\tau_1^1 < 0$ on that interval. The numerator of τ_2^1 is an increasing function of α, and we have the numerator negative when $\alpha \to 0$, and positive when $\alpha \to max(\alpha)$. Hence we know there exists an $\underline{\alpha_1}$ such that the numerator becomes positive. We also know that the numerator is positive at $\hat{\alpha_1}$, so $\underline{\alpha_1} < \hat{\alpha_1}$. When $\alpha \to \hat{\alpha_1}$, $\tau_2^1 \to +\infty$. Tehrefore, there exists an $\overline{\alpha_1}$ such that on $(\overline{\alpha_1}, \hat{\alpha_1})$, $\tau_2^1 > 1$. Results on the effect of taxation on consumption can be summarized in the following table:

α	c_{p1}^*
$0 < \alpha < \underline{\alpha_1}$	decreases
$\underline{\alpha_1} < \alpha < \overline{\alpha_1}$	has an inverted U-shape
$\overline{\alpha_1} < \alpha$	increases

  - When $c_0 > \overline{c_0}$, the discriminant is negative and $\alpha > \hat{\alpha_1}$, so the polynomial is negative, i.e c_{p1}^* increases.

For the worker

The derivative of c_{p2}^* with respect to τ is a second degree polynomial. Polluting consumption decreases for:

$$\tau^2 \left(K^*p^*(1-\alpha)\left(\frac{r^*/p^*}{\eta} - \eta\delta\right) + \left(\frac{r^*/p^*(1-\eta)}{\eta}\right)\frac{(1-\alpha)}{2\alpha(1-\varepsilon)} - \frac{2\eta c_0}{\alpha}\right) + \tau\left(\left(\frac{r^*/p^*(1-\eta)}{\eta}\right)\frac{(1-\alpha)}{\alpha(1-\varepsilon)} - \frac{4\eta c_0}{\alpha}\right) - \left(K^*p^*\left(\frac{r^*/p^*}{\eta} - \eta\delta - \frac{\left(\frac{r^*/p^*(1-\eta)}{\eta}\right)}{2\alpha(1-\varepsilon)}\right) + \frac{2\eta c_0}{\alpha}\right) > 0,$$

positive when $\left(2\left(\frac{r^*/p^*}{\eta} - \eta\delta\right) - \frac{\left(\frac{r^*/p^*(1-\eta)}{\eta}\right)}{1-\varepsilon}\right)\left(K^*p^*(\frac{r^*/p^*}{\eta} - \delta\eta)(1-\alpha) - 2\eta c_0\right) > 0.$

When this is negative, the sign of the polynomial is equal to the sign of $K^*p^*(1-\alpha)\left(\frac{r^*/p^*}{\eta} - \eta\delta + \left(\frac{r^*/p^*(1-\eta)}{\eta}\right)\frac{(1-}{2\alpha(}\right.$ $\frac{2\eta c_0}{\alpha}$, which is positive for $0 < \alpha < \hat{\alpha_2}$ and negative for $\hat{\alpha_2} < \alpha < 1$.

- $\varepsilon < \varepsilon^*$:

- When $c_0 < \overline{c_0}$, the discriminant is positive, so there are two roots:

$$\tau_1^2 = \frac{\frac{4\eta c_0}{\alpha} - \left(\frac{r^*/p^*(1-\eta)}{\eta}\right)\frac{(1-\alpha)}{\alpha(1-\varepsilon)} - \sqrt{2K^*p^*\left(2\left(\frac{r^*/p^*}{\eta}-\eta\delta\right) - \frac{\left(\frac{r^*/p^*(1-\eta)}{\eta}\right)}{1-\varepsilon}\right)\left(K^*p^*\left(\frac{r^*/p^*}{\eta}-\delta\eta\right)(1-\alpha)-2\eta c_0\right)}}{2\left(K^*p^*(1-\alpha)\left(\frac{r^*/p^*}{\eta}-\eta\delta+\left(\frac{r^*/p^*(1-\eta)}{\eta}\right)\frac{(1-\alpha)}{2\alpha(1-\varepsilon)}\right)-\frac{2\eta c_0}{\alpha}\right)}$$

$$\tau_2^2 = \frac{\frac{4\eta c_0}{\alpha} - \left(\frac{r^*/p^*(1-\eta)}{\eta}\right)\frac{(1-\alpha)}{\alpha(1-\varepsilon)} + \sqrt{2K^*p^*\left(2\left(\frac{r^*/p^*}{\eta}-\eta\delta\right) - \frac{\left(\frac{r^*/p^*(1-\eta)}{\eta}\right)}{1-\varepsilon}\right)\left(K^*p^*\left(\frac{r^*/p^*}{\eta}-\delta\eta\right)(1-\alpha)-2\eta c_0\right)}}{2\left(K^*p^*(1-\alpha)\left(\frac{r^*/p^*}{\eta}-\eta\delta+\left(\frac{r^*/p^*(1-\eta)}{\eta}\right)\frac{(1-\alpha)}{2\alpha(1-\varepsilon)}\right)-\frac{2\eta c_0}{\alpha}\right)}$$

  We know $a > 0$ for $\alpha < \hat{\alpha}_2$ and $a < 0$ for $\alpha > \hat{\alpha}_2$, with $\hat{\alpha}_2 < 0$ for $\varepsilon < 1 - \frac{K^*p^*B_2}{4\eta c_0}$. When $\varepsilon < 1 - \frac{K^*p^*B_2}{4\eta c_0}$, the numerators of τ_1^2 and τ_2^2 are always positive, so $\tau_1^2 < 0$ and $\tau_2^2 < 0$ for any value of α: c^*_{p2} increases.
  When $\varepsilon > 1 - \frac{K^*p^*B_2}{4\eta c_0}$, the numerator of τ_1^2 is an increasing function of α and is equal to zero at $\hat{\alpha}_2$, so $\tau_1^2 < 0$ for any value of α. The numerator of τ_2^2 has an inverted U shape in α and is negative when α is close to zero and positive around its maximal value, meaning there exists an $\underline{\alpha_2} < \hat{\alpha}_2$ such that the numerator becomes positive. When $\alpha \to \hat{\alpha}_2$, $\tau_2^2 \to +\infty$, meaning there exists $\overline{\alpha_2} < \hat{\alpha}$ such that $\tau_2^2 > 1$. Results can be summarized in the following table:

α	c^*_{p2}
$0 < \alpha < \underline{\alpha_2}$	decreases
$\underline{\alpha_2} < \alpha < \overline{\alpha_2}$	has an inverted U-shape
$\overline{\alpha_2} < \alpha$	increases

- When $c_0 > \overline{c_0}$, the discriminant is negative and $\alpha > \hat{\alpha}_2$ so c^*_{p2} increases.

\- $\varepsilon > \varepsilon^*$:

- When $c_0 < \overline{c_0}$, the discriminant is negative and $\alpha < \hat{\alpha}_2$ so the polynomial is negative, and c^*_{p2} decreases.
- When $c_0 > \overline{c_0}$, the discriminant is positive and there are 2 roots τ_1^2 and τ_2^2. In this case, we always have $\varepsilon > 1 - \frac{K^*p^*B_2}{4\eta c_0}$, i.e $a > 0$ for $\alpha < \hat{\alpha}_2$ and $a < 0$ otherwise. The numerator of τ_1^2 has an inverted-U shape in α, is negative at the minimum value of α and positive at $\alpha = 1$, so it crosses 0 once at $\hat{\alpha}$. Hence $\tau_1^2 < 0$ always. The numerator of τ_2^2 is an increasing function of α, negative at $min(\alpha)$ and positive at $\alpha = 1$. As before, there exists an $\underline{\alpha_2} < \hat{\alpha}_2$ such that the numerator becomes positive, and an $\overline{\alpha_2} < \hat{\alpha}_2$ such that $\tau_2^2 > 1$. Results are summarized in the following table:

α	c^*_{p2}
$0 < \alpha < \underline{\alpha_2}$	decreases
$\underline{\alpha_2} < \alpha < \overline{\alpha_2}$	has an inverted U-shape
$\overline{\alpha_2} < \alpha$	increases

Proof of Proposition 6

Impact on overall consumption

For the capitalist: Taking the derivative of C_1^* with respect to τ gives the following:

$$\tau^{c2}c_0(1-\alpha)(2\varepsilon-1+\alpha)+2\tau c_0(2\varepsilon-1+\alpha)+K^*p^*(\frac{r^*/p^*}{\eta}-\delta)\alpha\varepsilon+c_0(2\varepsilon-1)$$

Solving for the polynomial gives that $\frac{\partial C_1^*}{\partial\tau}>0$ for $c_0<\overline{c_0}$ and $\varepsilon<\frac{1-\alpha}{2}$ or $c_0>\overline{c_0}$ and $\varepsilon>\frac{1-\alpha}{2}$.

- $c_0<\overline{c_0}$:
  If $\varepsilon>\frac{1-\alpha}{2}$, C_1^* increases.
  If $\varepsilon<\frac{1-\alpha}{2}$, the polynomial as two roots:

  $$\tau_{11}^c=\frac{2c_0(1-\alpha-2\varepsilon)-\sqrt{\Delta_1}}{2c_0(1-\alpha)(2\varepsilon-1+\alpha)}\tau_{12}^c=\frac{2c_0(1-\alpha-2\varepsilon)+\sqrt{\Delta_1}}{2c_0(1-\alpha)(2\varepsilon-1+\alpha)}$$

  and Δ_1 the discriminant associated to $\frac{\partial C_1^*}{\partial\tau}$. $\tau_{12}^c<0$ as the nominator is positive and the denominator negative.
  $\tau_{11}^c>0$ iff $\varepsilon>\frac{\eta c_0}{2\eta c_0+K^*p^*\alpha\left(\frac{1-\beta_1}{\beta_1}+\delta(1-\eta)\right)}\equiv\varepsilon_1$. If not, $\tau_{11}^c<0$ and C_1 increases. Assuming $\varepsilon_1<\varepsilon<\frac{1-\alpha}{2}$, we must now ensure that $\tau_{11}^c<1$.

  $$\tau_{11}^c<1\Longleftrightarrow\alpha^2\eta c_0+\alpha\left(2\varepsilon\eta c_0-K^*p^*\varepsilon\left(\frac{1-\beta_1}{\beta_1}+\delta(1-\eta)\right)-4\eta c_0\right)+4\eta c_0(1-2\varepsilon)>0$$

  The polynom in α has two roots, $\hat{\alpha}$ and $\hat{\hat{\alpha}}$. $\hat{\hat{\alpha}}<0$ and $0<\hat{\alpha}<1$ iff $\varepsilon>\frac{\eta c_0}{6\eta c_0+K^*p^*\left(\frac{1-\beta_1}{\beta_1}+\delta(1-\eta)\right)}$, which is always verified under the assumption $\varepsilon>\varepsilon_1$. Hence, we have $\tau_{11}^c<1$ for $\alpha<\hat{\alpha}$ and C_1^* has an inverted U-shape, and $\tau_{11}^c>1$ for $\alpha>\hat{\alpha}$, leading to an increase in C_1^*.

- $c_0>\overline{c_0}$:
  If $\varepsilon<\frac{1-\alpha}{2}$, C_1^* decreases.
  $\varepsilon>\frac{1-\alpha}{2}$, the polynomial has two roots $\tau_{11}^c<0$ and τ_{12}^c. $\tau_{12}^c>0$ iff $\varepsilon<\varepsilon_1$. If $\varepsilon>\varepsilon_1$, C_1^*

  decreases.
  Assuming $\varepsilon<\varepsilon_1$ and after some computations, we find that $\tau_{12}>1$, meaning that C_1^* increases.

For the worker: Taking the derivative of C_2^* with respect to τ gives:

$$\tau^{c2}c_0(1-\alpha)(2(1-\varepsilon)-1+\alpha)+2\tau c_0(2(1-\varepsilon)-1+\alpha)+K^*p^*(\frac{r^*/p^*}{\eta}-\delta)\alpha(1-\varepsilon)+c_0(1-2\varepsilon)$$

Solving for the polynomial gives that $\frac{\partial C_2^*}{\partial\tau}>0$ for $c_0<\overline{c_0}$ and $\varepsilon>\frac{1+\alpha}{2}$ or $c_0>\overline{c_0}$ and $\varepsilon<\frac{1+\alpha}{2}$.

- $c_0<\overline{c_0}$:

If $\varepsilon < \frac{1+\alpha}{2}$, C_2^* increases.
If $\varepsilon > \frac{1-\alpha}{2}$, the polynomial as two roots:

$$\tau_{21}^c = \frac{2c_0(1-\alpha-2(1-\varepsilon)) - \sqrt{\Delta_2}}{2c_0(1-\alpha)(2(1-\varepsilon)-1+\alpha)} \tau_{22}^c = \frac{2c_0(1-\alpha-2(1-\varepsilon)) + \sqrt{\Delta_2}}{2c_0(1-\alpha)(2(1-\varepsilon)-1+\alpha)}$$

and Δ_2 the discriminant associated to $\frac{\partial C_2^*}{\partial \tau}$. $\tau_{22}^c < 0$ as the nominator is positive and the denominator negative, and after some computations we find $\tau_{21}^c > 1$. Hence, C_2^* decreases.

- $c_0 > \overline{c_0}$:
  If $\varepsilon > \frac{1+\alpha}{2}$, C_1^* decreases.
  If $\varepsilon < \frac{1+\alpha}{2}$, the polynomial has two roots τ_{21}^c and τ_{12}^c tht are both negative. Therefore C_2^*

  increases.

Impact on welfare

An increase in the commodity tax rate leads to changes in utilities:

$$\frac{dU_i}{d\tau} = \mu \frac{dE^*}{d\tau} E^{*\mu-1} \frac{C_i^{*1-\sigma}}{1-\sigma} + \frac{dC_i}{d\tau} C_i^{*-\sigma} E^{*\mu}$$

The result is straightforward:

- when $c_0 < \overline{c_0}$, if C_i increases, then U_i increases. If C_i decreases, U_i increases if $\mu > \mu_i^*(\tau, \varepsilon)$ and decreases otherwise;
- when $c_0 > \overline{c_0}$, if C_i decreases, then U_i decreases. If C_i increases, U_i decreases if $\mu > \mu_i^*(\tau, \varepsilon)$ and increases otherwise.

Proof of Proposition 7

Looking at the characteristic polynomial:

$$P(-1) = 2(1+D) - \frac{\frac{\beta_1}{p} C_1^{*1-\sigma} K^* f''(K^*)(1+\tau)^\alpha p^{1-\alpha}}{\alpha^\alpha (1-\alpha)^{1-\alpha} p \left(1 + \frac{\alpha\varepsilon\tau}{1+\tau(1-\alpha)}\right) \left(-\sigma C_1^{*-\sigma} - \mu\alpha C_1^{*1-\sigma} \frac{(1+\tau)^\alpha p^{1-\alpha}}{\alpha^\alpha(1-\alpha)^{1-\alpha}}\right)} > 0$$

$$P(0) = D > 0$$

$$P(1) = \frac{\frac{\beta_1}{p} C_1^{*1-\sigma} K^* f''(K^*)(1+\tau)^\alpha p^{1-\alpha}}{\alpha^\alpha (1-\alpha)^{1-\alpha} p \left(1 + \frac{\alpha\varepsilon\tau}{1+\tau(1-\alpha)}\right) \left(-\sigma C_1^{*-\sigma} - \mu\alpha C_1^{*1-\sigma} \frac{(1+\tau)^\alpha p^{1-\alpha}}{\alpha^\alpha(1-\alpha)^{1-\alpha}}\right)} < 0$$

When $p < \overline{p}$, $D > 1$ and $T > 2$ so we have a saddle.
When $p > \overline{p}$, $D > 1$ if $\mu > \overline{\mu}$ and lower than 1 otherwise. When $D < 1$, $T < 2$ whenever $\mu < \mu^* < \overline{\mu}$. in any case, $P(1) < 0$ so we have a saddle.

Proof of Proposition 8

Looking at the characteristic polynomial:

$$
\begin{aligned}
P(-1) &= 2(1+D) - \frac{\beta_1 f''(K^*)C_1^*(1+\tau)^\alpha p^{1-\alpha}}{\alpha^\alpha(1-\alpha)^{1-\alpha}\left(1+\frac{\alpha\varepsilon\tau}{1+\tau(1-\alpha}\right)(\sigma C_1^* - (1+\tau)c_0)} \\
P(0) &= D \\
P(1) &= \frac{\beta_1 f''(K^*)C_1^*(1+\tau)^\alpha p^{1-\alpha}}{\alpha^\alpha(1-\alpha)^{1-\alpha}\left(1+\frac{\alpha\varepsilon\tau}{1+\tau(1-\alpha}\right)(\sigma C_1^* - (1+\tau)c_0)}
\end{aligned}
$$

If $c_0 < c_0^*$, then $\sigma C_1^* - (1+\tau)c_0 > 0$ and we have a saddle.
If $c_0 > c_0^*$, $P(1) > 0$ so we must check the sign of $P(-1)$ and $P(0)$ in order to know whether there is indeterminacy of the equilibrium or not.
$D > 0$ always so $P(0) > 0$. Looking at $P(-1)$, we have $P(-1) > 0$ if $c_0 > c_0^{**}$, meaning the equilibrium is either a sink or a source, depending on whether $D > 1$ or not, meaning whether $p < \overline{p}$, as for proposition 7. As $\mu = 0$ here, $D < 1$ if $p > \overline{p}$.

Proof of Proposition 9

Looking at the characteristic polynomial:

$$
\begin{aligned}
P(-1) &= 2(1+D) - \frac{\frac{\beta_1}{p}C_1^{*1-\sigma}K^* f''(K^*)(1+\tau)^\alpha p^{1-\alpha}}{\alpha^\alpha(1-\alpha)^{1-\alpha}p\left(1+\frac{\alpha\varepsilon\tau}{1+\tau(1-\alpha)}\right)\left(+\sigma C_1^{*-\sigma} + \mu\alpha C_1^{*1-\sigma}\frac{(1+\tau)^\alpha p^{1-\alpha}}{\alpha^\alpha(1-\alpha)^{1-\alpha}} - (1-\tau)c_0 C_1^{*-\sigma-1}\right)} \\
P(0) &= D \\
P(1) &= \frac{\frac{\beta_1}{p}C_1^{*1-\sigma}K^* f''(K^*)(1+\tau)^\alpha p^{1-\alpha}}{\alpha^\alpha(1-\alpha)^{1-\alpha}p\left(1+\frac{\alpha\varepsilon\tau}{1+\tau(1-\alpha)}\right)\left(\sigma C_1^{*-\sigma} + \mu\alpha C_1^{*1-\sigma}\frac{(1+\tau)^\alpha p^{1-\alpha}}{\alpha^\alpha(1-\alpha)^{1-\alpha}} - (1-\tau)c_0 C_1^{*-\sigma-1}\right)}
\end{aligned}
$$

As $\sigma C_1^{*-\sigma} + \mu\alpha C_1^{*1-\sigma}\frac{(1+\tau)^\alpha p^{1-\alpha}}{\alpha^\alpha(1-\alpha)^{1-\alpha}} - (1-\tau)c_0 C_1^{*-\sigma-1} < 0$, $P(1) < 0$ always.

$P(-1) < 0$ for $\mu < \tilde{\mu}$, in which case $D < -1$ which gives a source. Otherwise, $P(-1) > 0$ and hence a saddle.

References

[1] Antoci A., Galeotti M., and Russu P. Consumption of private goods as substitutes for environmental goods in an economic growth model. *Nonlinear Analysis: Modelling and Control*, 10(1):3–34, 2005.

[2] William J. Baumol. On taxation and the control of externalities. *The American Economic Review*, 62(3):307–322, 1972.

[3] Robert A. Becker. On the long-run steady state in a simple dynamic model of equilibrium with heterogeneous households. *The Quarterly Journal of Economics*, 95(2):375–382, 1980.

[4] Robert A Becker and Ciprian Foias. A characterization of ramsey equilibrium. *Journal of Economic Theory*, 41(1):173–184, 1987.

[5] Alexandre Berthe and Luc Elie. Mechanisms explaining the impact of economic inequality on environmental deterioration. *Ecological Economics*, 116:191–200, 2015.

[6] Lawrence H. Goulder, Marc A.C. Hafstead, GyuRim Kim, and Xianling Long. Impacts of a carbon tax across us household income groups: What are the equity-efficiency trade-offs? *Journal of Public Economics*, 175:44 – 64, 2019.

[7] Marjan W. Hofkes. Environmental policies. *Environmental and Resource Economics*, 20(1):1–26, 2001.

[8] Jun ichi Itaya. Can environmental taxation stimulate growth? the role of indeterminacy in endogenous growth models with environmental externalities. *Journal of Economic Dynamics and Control*, 32(4):1156–1180, 2008.

[9] David Klenert, Gregor Schwerhoff, Ottmar Edenhofer, and Linus Mattauch. Environmental taxation, inequality and engel's law: The double dividend of redistribution. *Environmental and Resource Economics*, 71(3):605 – 624, 2018.

[10] A Lans Bovenberg and Ruud A de Mooij. Environmental tax reform and endogenous growth. *Journal of Public Economics*, 63(2):207–237, 1997.

[11] Liptak B. G. Liu, D. H. F. *Wastewater treatment.* CRC Press, Boca Raton, 2000.

[12] Walid Oueslati. Growth and welfare effects of environmental tax reform and public spending policy. *Economic Modelling*, 45:1–13, 2015.

[13] A.C Pigou. *The Economics of Welfare.* Macmillan, London, 1920.

[14] Sebastian Rausch and Giacomo A. Schwarz. Household heterogeneity, aggregation, and the distributional impacts of environmental taxes. *Journal of Public Economics*, 138:43–57, 2016.

[15] Domenico Scalera. Optimal consumption and the environment choosing between 'clean'and 'dirty'goods. *Environmental and Resource Economics*, 7(4):375–389, 1996.

[16] Gerhard Sorger. On the structure of ramsey equilibrium: Cycles, indeterminacy, and sunspots. *Economic Theory*, 4(5):745–764, 1994.