Stable partitions for proportional generalized claims problems*

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Abstract

We consider a set of agents, e.g., a group of researchers, who have claims on an endowment, e.g., a research budget from a national science foundation. The research budget is not large enough to cover all claims. Agents can form coalitions and coalitional funding is proportional to the sum of the claims of its members, except for singleton coalitions which do not receive any funding. We analyze the structure of stable partitions when coalition members use well-behaved rules to allocate coalitional endowments, e.g., the well-known constrained equal awards rule (CEA) or the constrained equal losses rule (CEL).

For continuous, (strictly) resource monotonic, and consistent rules, stable partitions with (mostly) pairwise coalitions emerge. For CEA and CEL we provide algorithms to construct such a stable pairwise partition. While for CEL the resulting stable pairwise partition is assortative and sequentially matches lowest claims pairs, for CEA the resulting stable pairwise partition is obtained sequentially by matching in each step either a highest claims pair or a highest-lowest claims pair.

More generally, we can also assume that the minimal coalition size to have a positive endowment is $\theta \geq 2$. We then show how all results described above are extended to this general case.

Keywords: claims problems, coalition formation, stable partitions.

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Extended abstract

The formation of coalitions is a pervasive aspect of social, economic, or political environments. Agents form coalitions in very different situations in order to achieve some joint benefits. Cooperation between agents is sometimes hampered by the existence of two opposing fundamental forces: on the one hand, the increasing returns to scale, which incentivizes agents to cooperate and, therefore, to form large coalitions and, on the other hand, the heterogeneity of agents, which causes instability and pushes towards the formation of only small coalitions.

Gallo and Inarra (2018) introduce *generalized claims problems*[1] to deal with coalition formation in a bankruptcy framework. A generalized claims problem consists of a group of agents, each of them with a claim and a set of *coalitional endowments*, one for each possible coalition, which are not sufficient to meet the claims of their members. Coalitional endowments are divided among their members according to a pre-specified rule, which thus is a decisive element of the coalition formation process. Their main result (Gallo and Inarra, 2018, Theorem 2) states that, given a generalized claims problem, there is a stable partition for each coalition formation problem that is induced by a continuous rule if and only if the continuous rule satisfies resource monotonicity and consistency. In this paper, we study the structure of stable partitions under different resource monotonic and consistent rules to answer two types of questions: What coalition sizes can emerge? & Who are the coalition partners?

The model proposed by Gallo and Inarra (2018) does not impose any restriction on coalitional endowments and consequently answering the above questions is not really possible in their general model. In contrast, we first consider *non-singleton proportional generalized claims problems* where singleton coalitions have zero endowments and all remaining coalitional endowments are a fixed proportion of the sum of their members' claims. Proportionality is justified in many situations such as the funding of research projects where the budgets are often divided proportionally to funding needs or according to other funding criteria such as project quality.[2] Moreover, in many situations, institutions are interested to spark cooperation and hence, discourage singleton coalitions. Note that for didactic reasons we first focus on non-singleton

[1]Note that these authors use the term *coalition formation problem with claims* instead of *generalized claims problems*.

[2]Other examples can be found in a bankruptcy situation where assets have to be allocated proportionally among creditors according to their claims or, in a legislature, where seats are distributed proportionally among the parties according to voting shares.

proportional generalized claims problems and later on extend our results to address minimal coalition sizes $\theta > 2$ for positive coalitional endowments.

Non-singleton proportional generalized claims problems are a subclass of the class of generalized claims problems studied by Gallo and Inarra (2018) and hence their results hold. Then, given a non-singleton proportional generalized claims problem, we first characterize the structure of any possible stable partition when the rule applied satisfies continuity, strict resource monotonicity, and consistency. We show that at most one singleton coalition belongs to each stable partition and that for each coalition in the stable partition with size larger than two, each agent of the coalition receives a proportional payoff (Theorem 1). Furthermore, considering resource monotonicity instead of its strict version, even though we do not characterize all stable partitions, we show that a stable partition formed by pairwise coalitions (i.e., coalitions of size two) exists, with the exception of at most one singleton coalition if the set of agents is odd (Theorem 2).

With the result of Theorem 2 as the departure point, we analyze how agents sort themselves into pairwise coalitions under some parametric rules (see Young, 1987; Stovall, 2014). These parametric rules are well-studied in the literature because the payoff of each agent is given by a function that depends only on the claim of the agent and a parameter that is common to all agents. We focus on two well-known parametric rules that represent two egalitarian principles: the constrained equal awards rule (CEA) and the constrained equal losses rule (CEL). On the one hand, CEA divides the endowment as equally as possible subject to no agent receiving more than her claim (e.g., rationing toilet paper when shortage occurs). On the other hand, CEL divides the endowment as equally as possible subject to no agent receiving a negative amount (e.g., equal sacrifice taxation when utility is measured linearly[3]).

We propose two algorithms, one for each rule, to find a stable and (almost) pairwise partition. The *CEA algorithm* sequentially pairs off either two highest-claim agents or a highest-claim with a lowest-claim agent (Theorem 3). Examples of the first type of cooperation are found in social environments where agents tend to join other agents with similar characteristics. In contrast, the second type of cooperation may be interpreted as a transfer of knowledge between agents as happens, for instance, between

[3]The idea of the equal sacrifice principle in taxation is that all tax payers end up sacrificing equally, according to some cardinal utility function. Young (1988) provides a characterization of the family of equal-sacrifice rules based on a few compelling principles and, more recently, Chambers and Moreno-Ternero (2017) generalize the previous family.

apprentices and advisors. While the CEA algorithm produces stable partitions that can contain *assortative* as well as *extremal pairwise coalitions*, the *CEL algorithm* is purely assortative and sequentially pairs off lowest claims agents (Theorem 4).

Next, we consider the fact that some funding calls may require a larger minimal number of agents in a group to generate a positive endowment. Therefore, we introduce another subclass of the class of generalized claims problems, *θ-minimal proportional generalized claims problems*. In these problems, coalitions of size lower than θ have zero endowments and all remaining coalitional endowments are a fixed proportion of the sum of their members' claims. We generalize our results from $\theta = 2$, the non-singleton proportional generalized claims problems, to any $\theta \in \mathbb{N}$. More specifically, we first show that when the rule applied satisfies continuity, strict resource monotonicity, and consistency, then there are fewer than θ agents in coalitions of size smaller than θ and that for each coalition in the stable partition with size larger than θ, each agent of the coalition receives a proportional payoff (Theorem 5). Moreover, if the rule satisfies resource monotonicity instead of its strict version, we show that a stable partition formed by the maximal possible number of coalitions of size θ and one coalition (of size lower than θ) formed by the remaining agents exists (Theorem 6).

In a similar way as in the non-singleton model, with the result of Theorem 6 as the departure point, we analyze how agents sort themselves into coalitions of size θ under the CEA and the CEL rules. We propose two algorithms, one for each rule, to find a stable partition. The *θ-CEA algorithm* generates a partition formed by coalitions of size θ constructed by sequentially adding a lowest-claim agent or a highest-claim agent (Theorem 7). For the CEL rule, an assortative stable partition is obtained by sequentially pairing off θ lowest-claim agents (Theorem 8).

There is a large number of papers that pay attention to the structure of the coalitions that form. Becker (1973) and Greenberg and Weber (1986) introduce the notion of assortative coalitions.[4] Observe that in both our algorithms assortative coalitions (in terms of claims) may form.

Finally, there are many papers dealing with assortative stable coalitions. We briefly discuss three of them.

Barberà et al. (2015) consider a model in which each agent is endowed with a

[4] Assortativeness is based on an ordering of agents according to a specific variable such as claims, productivity, or location. Alternative terminology includes that of consecutive coalitions.

productivity level and agents can cooperate to perform certain tasks. Each coalition generates an output equal to the sum of its members' productivities. The authors then analyze the formation of coalitions when all agents in a society vote between meritocracy and egalitarianism. They find societies where assortative and non-assortative partitions (in terms of productivity) arise.

In a bargaining framework, Pycia (2012) presents a model in which each agent has a utility function and, for each possible coalition of agents, there is an output to be distributed among its members. He analyzes coalition formation games induced by different bargaining rules and shows that when agents are endowed with productivity levels and "when shares are divided by a stability-inducing sharing rule, agents sort themselves into coalitions in a predictably assortative way". Pycia (2012) deals with many-to-one problems and his notion of assortativeness implies that the most productive agents join the most productive firms.

Finally, Bogomolnaia et al. (2008) study societies where agents are located in an interval and form jurisdictions to consume public projects, which are located in the same interval. Agents share their costs equally and they divide transportation costs to the location of the public project based on its distance to each agent. They analyze both core and Nash stable partitions with a focus on assortative and non-assortative (in terms of location) stable jurisdiction structures.

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