# The Neoclassical Growth Model with Time-Inconsistent Decision Making and Perfect Foresight

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**Abstract**

In this paper, we propose an approach to describe the behavior of naive agents with quasi-hyperbolic discounting in the neoclassical growth model. To study time-inconsistent decision making of an agent who cannot commit to future actions, we introduce the notion of sliding equilibrium and distinguish between pseudo-perfect foresight and perfect foresight. The agent with pseudo-perfect foresight revises both the consumption path and expectations about prices; the agent with perfect foresight correctly foresees prices in a sliding equilibrium and is naive only about their time inconsistency. We prove the existence of sliding equilibria for the class of isoelastic utility functions and show that generically consumption paths are not the same under quasi-hyperbolic and exponential discounting. Observational equivalence only holds in the well-known cases of a constant interest rate or logarithmic utility. Our results suggest that perfect foresight implies a higher long-run capital stock and consumption level than pseudo-perfect foresight.

**Keywords**: Quasi-hyperbolic discounting; Observational equivalence; Time inconsistency; Naive agents; Sliding equilibrium; Perfect foresight

**JEL Classification**: D15, D91, E21, O40

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# 1 Introduction

The aim of this paper is to study the neoclassical growth model with quasi-hyperbolic discounting where the agent is naive about their time inconsistency, but can have perfect foresight about future economic conditions. The literature on the Ramsey model with quasi-hyperbolic discounting mainly focuses on sophisticated agents and studies time-consistent consumption paths. While time consistency is an appealing criterion of rationality, real-world decisions are often time-inconsistent, which necessitates rigorous analysis of naive behavior.

To study time-inconsistent decision making, we employ a distinct equilibrium concept that we label **sliding equilibrium**. This concept allows to describe the behavior of a naive agent who revises their consumption path at each date. Sliding equilibria critically depend on the formation of expectations, which is typically overlooked in the characterization of naive decisions. We clarify the role of expectations by distinguishing between pseudo-perfect foresight and perfect foresight. An agent with pseudo-perfect foresight at each date revises both their consumption path and expectations about prices, while an agent with perfect foresight correctly foresees prices on a sliding equilibrium path and is naive only about their time inconsistency.

Our study derives the conditions under which consumption paths are the same under quasi-hyperbolic and exponential discounting, i.e., under which sliding equilibrium paths are observationally equivalent to the optimal paths in a standard Ramsey model. While previous literature emphasizes the prevalence of observational equivalence, we prove that it is not a generic phenomenon and occurs only in very special cases. Our results imply that under naivety, quasi-hyperbolic discounting almost always matters for saving behavior.

Standard economic growth models employ the assumption of exponential discounting. While analytically tractable and convenient, this assumption is not supported empirically. A large number of laboratory and field studies of time preferences show that discount rates are higher in the short run than in the long run, i.e., individuals exhibit a present bias (see, e.g., Ainslie, 1992; Frederick et al., 2002; DellaVigna, 2009). A well-known way to model present bias is to assume time-declining (hyperbolic) discounting, as opposed to constant (exponential) discounting (see, e.g., Angeletos et al., 2001, for a discussion and literature review).

The inherent property of hyperbolic discounting is time inconsistency. Strotz (1955) addresses the question of what will happen if an agent revises their optimal path at each date. He shows that the optimal path as viewed from any future date is a truncation of the original optimal path only for an agent with exponential discounting. An agent with another type of discounting (e.g., hyperbolic) is time-inconsistent and has a new optimal path at each date. Strotz discusses two strategies that can restore time consistency: precommitment where the agent commits to follow the original optimal path, and consistent planning where the agent chooses the best path among those that would be actually followed.

Pollak (1968) clarifies the ideas of Strotz and distinguishes between two types of agents who cannot commit to their future actions. First, he introduces a naive agent, who is unaware of their time-inconsistent preferences. A naive agent revises their optimal path at each date, and the outcome of this procedure is referred to as a "naive path".[1] Second, he introduces a sophisticated agent, who recognizes their time inconsistency. The path obtained under the strategy of consistent planning is referred to as a "sophisticated path". Pollak proves that in a cake-eating model with log-utility, the naive and sophisticated paths coincide.

Phelps and Pollak (1968) compare naive and sophisticated paths in a model with quasi-hyperbolic discounting, isoelastic utility and production technology with a constant marginal productivity of capital (i.e., an exogenous and constant interest rate). They note that a sophisticated path is a Nash equilibrium in a game between different generations. They characterize naive and sophisticated paths for general isoelastic utility, and show that under log-utility, both paths coincide.[2]

Following Laibson (1997), most of the literature on quasi-hyperbolic discount-

[1]The concept of a naive path resembles the notion of a sliding path or a rolling plan (see, e.g., Goldman, 1968; Kaganovich, 1985).

[2]Sophisticated paths in the spirit of Strotz are further studied by Peleg and Yaari (1973) and Goldman (1980).

ing is concerned with sophisticated agents. An agent is modeled as a sequence of autonomous temporal selves with conflicting preferences, whose behavior is described by a dynamic game played among the agent's different selves. Sophisticated paths correspond to the equilibria of such a game, which are typically refined to symmetric Markov perfect Nash equilibria.[3]

Many recent studies of the effects of hyperbolic discounting on consumption and savings decisions within a neoclassical growth model employ the assumption of sophisticated agents (see, e.g., Harris and Laibson, 2001; Krusell and Smith, 2003; Maliar and Maliar, 2006; Ekeland and Lazrak, 2010). It is common to analyze sophisticated paths either under log-utility or under the assumption of an exogenous and constant interest rate, primarily because both assumptions simplify the analysis and allow for analytical solutions.[4] In particular, Krusell et al. (2002) provide a closed-form solution for the Markov perfect equilibrium in the Ramsey model with quasi-hyperbolic discounting, log-utility and Cobb–Douglas production technology. Under a constant interest rate, a number of important results are obtained, even for general isoelastic utility (see, e.g., Bernheim et al., 2015; Cao and Werning, 2018).

The literature on growth models with naive agents is much less prolific.[5] In a seminal contribution, Barro (1999) studies the Ramsey model with time-declining discounting and log-utility. He provides a full solution for an agent who revises their consumption path at each instant of time, i.e., he implicitly assumes a naive agent and obtains a naive path.[6] Barro was also one of the first to note that the observable outcome of consumption and savings decisions made by agents with hyperbolic discounting does not necessarily differ from that of agents with expo-

[3]The properties of such equilibria in a very general setting are studied by Sorger (2004).

[4]Each of these assumptions allows also to study sophisticated paths under varying levels of commitment (see Sorger, 2007), as well as for heterogeneous in their time preferences agents (see Drugeon and Wigniolle, 2019).

[5]A notable recent exception is Ahn et al. (2020) who develop a general axiomatic theory of naivety and apply it to a consumption-savings problem with constant interest rate.

[6]Barro implicitly considers a naive agent, but since under log-utility the propensity to consume is constant, his solution looks like a time-consistent one. Perhaps, this is the reason why many subsequent authors mistook Barro's naive agent for a sophisticated agent.

nential discounting. A number of models (e.g., Laibson, 1996; Barro, 1999; Krusell et al., 2002) show that the optimal paths of consumption and capital can coincide under hyperbolic and exponential discounting. This phenomenon is discussed under the header of **observational equivalence**. Barro (1999) shows that a naive path under log-utility and a concave production function is observationally equivalent to an optimal path in the standard Ramsey model.

The results by Barro (1999) are extended by Findley and Caliendo (2014), who consider a finite-horizon model with quasi-hyperbolic discounting and an exogenous interest rate. They prove that for any fixed and constant planning horizon, a naive path is observationally equivalent to the optimal path in a model with exponential discounting in two cases: i) for log-utility and ii) for general isoelastic utility and a constant interest rate.[7]

The question of which type of agents, naive or sophisticated, is more appropriate in the context of growth theory, is beyond the scope of this paper. There are supporting arguments for both approaches: it is suggested that naivety is closer to real-world decision making, while sophistication is more consistent with the standard notion of rationality. In the prior literature, important results about both naive and sophisticated paths are obtained either under log-utility (in which case, roughly speaking, nothing depends on the interest rate), or under an exogenous and constant interest rate. Even under these simplifying assumptions, a sophisticated path is a complicated game-theoretic notion, and its analysis is technically demanding.[8] In this paper, we are interested in studying time-inconsistent decision making under fairly general assumptions. We consider a naive agent and characterize their behavior in a general equilibrium framework with an endogenously changing interest rate and isoelastic utility.

It is well known that the standard Ramsey model with exponential discounting

[7]Farzin and Wendner (2014) in a similar model confirm these results and note that hyperbolic discounting and short-term planning imply a hump-shaped dynamics of saving rate which is consistent with empirical evidence.

[8]For instance, much less is known about observational equivalence of sophisticated paths. Under log-utility, observational equivalence holds for a linear production technology (see Phelps and Pollak, 1968), as well as for a Cobb–Douglas production technology (see Krusell et al., 2002).

exhibits two properties. First, the optimal path of consumption and capital is time-consistent, i.e., re-planning at any date $\tau$ does not change the optimal path from $\tau$ onwards, so the optimal path as viewed from date $\tau$ (date-$\tau$ optimal path) is a truncation of the original date-0 optimal path. Second, the optimal path can be decentralized as an equilibrium path. In equilibrium, an agent who perfectly foresees interest and wage rates chooses the path of consumption and savings (capital) that coincides with the optimal one.

However, in the Ramsey model with quasi-hyperbolic discounting where an agent cannot commit to future actions and is naive about their time inconsistency, both properties generically fail to hold. First, an optimal path is time-inconsistent. That is, a date-$\tau$ optimal path differs from the truncation of a date-0 optimal path. Second, the traditional general equilibrium logic according to which optimal and equilibrium paths are essentially the same does not apply, because, due to time inconsistency, the notion of "perfect foresight" in equilibrium is unclear.

To address the first issue (time inconsistency of an optimal path), we focus on **sliding optimal paths**. A sliding optimal path consists of only the date-$\tau$ choices of the date-$\tau$ optimal paths of consumption and capital. For each $\tau \geq 0$, new date-$\tau$ optimal path is obtained, and of each of those paths, the sliding optimal path picks only the date-$\tau$ elements. We characterize its properties and study observational equivalence, i.e., whether there is a discount factor for which the optimal path in the standard Ramsey model coincides with a sliding optimal path. We note that, to the best of our knowledge, observational equivalence for sliding optimal paths holds only in two cases: the stationary case and the case with log-utility and Cobb–Douglas production technology.

To address the second issue (the role of expectations) and to deal with time inconsistency of an equilibrium path, we focus on **sliding equilibrium paths** (cf. Borissov, 2013). Clearly, every date-$\tau$ path of consumption and savings (capital) chosen by an agent depends on expectations about future interest and wage rates, and so does every sliding equilibrium path.

We distinguish between two types of expectations in equilibrium: **pseudo-**

**perfect foresight** and **perfect foresight**. Under pseudo-perfect foresight, at each date $\tau$, an agent expects those interest and wage rates that occur on a date-$\tau$ equilibrium path, as if this path would be followed for all future dates $t > \tau$. Due to naivety, at each $t > \tau$, the agent revises their consumption path, and therefore also revises expectations of future equilibrium interest and wage rates. A **sliding equilibrium path under pseudo-perfect foresight** (which picks only the date-$\tau$ elements of each date-$\tau$ equilibrium path) is a decentralization of a sliding optimal path. Therefore, the results about observational equivalence of sliding equilibrium paths under pseudo-perfect foresight are the same as those of sliding optimal paths.

An agent with pseudo-perfect foresight is naive about both their future preferences and the future prices. While the first source of naivety is internal and can be explained by psychological factors (e.g., temptation), the second source of naivety is external and concerns general economic conditions. This distinction motivates the introduction of another type of expectations, namely, perfect foresight. An agent with perfect foresight at each date expects those interest and wage rates that occur on the resulting **sliding equilibrium path under perfect foresight**. Hence perfect foresight captures the case of a partially naive agent who remains unaware of their time-inconsistent preferences, but correctly foresees prices.[9]

We prove that for a general isoelastic utility a sliding equilibrium path under perfect foresight exists, and study the question of observational equivalence, i.e., whether a consumption path on a sliding equilibrium path under perfect foresight coincides with that in the standard Ramsey model. We show that observational equivalence holds in two cases: the stationary case and the log-utility case. Furthermore, we prove that these are the only cases where observational equivalence holds — within the class of isoelastic utility functions, there is **no observational equivalence** for sliding equilibrium paths under perfect foresight beyond the standard cases of log-utility and a constant interest rate.

[9]Following O'Donoghue and Rabin (2001), the literature on partial naivety typically assumes that an agent recognizes their present bias, but underestimates its impact. Our paper contributes to the discussion of partial naivety by noting that in a general equilibrium framework an agent can have perfect foresight about prices, but can be naive about their time inconsistency.

We also compare sliding equilibria under pseudo-perfect and perfect foresight in terms of long-run macroeconomic variables for both the stationary case and the case with log-utility and Cobb–Douglas production technology. Our results suggest that perfect foresight implies a higher saving rate and a higher long-run consumption level than pseudo-perfect foresight.

The remaining paper is organized as follows. Section 2 provides simple examples motivating and illustrating the subsequent analysis. In Section 3 we consider a sliding optimal path and its decentralization, a sliding equilibrium path under pseudo-perfect foresight. Section 4 introduces the central object of our study, a sliding equilibrium path under perfect foresight, and presents our main results. Section 5 concludes. The Appendix contains proofs and mathematical details supporting the analysis in the main text.

# 2 Motivational examples and useful facts

## 2.1 Motivational examples

Consider an infinitely lived agent with quasi-hyperbolic ($\beta$–$\delta$) discounting. Their (intertemporal) utility at each date $\tau$ is given by $u(c_\tau)+\beta\sum_{t=\tau+1}^{\infty}\delta^{t-\tau}u(c_t)$, where $0<\beta<1$ is the present bias parameter, $0<\delta<1$ is the long-run discount factor, and $u(c)$ is an isoelastic instantaneous utility function: $u(c)=c^{1-\rho}/(1-\rho)$ for $\rho>0$, with the convention that $\rho=1$ refers to the logarithmic case $u(c)=\ln c$. The agent cannot commit to future actions and is naive (unaware) of their time-inconsistent preferences.

At date $\tau$ the agent maximizes their utility under the intertemporal budget constraint

$$\sum_{t=0}^{\infty}\frac{c_{\tau+t}}{(1+r)^t} \;\le\; (1+r)s^*_{\tau-1}+\sum_{t=0}^{\infty}\frac{w}{(1+r)^t}\,,$$

where the initial savings $s^*_{\tau-1}$ and **constant** interest and wage rates, $r$ and $w$, are taken as given by the agent.[10]

[10]This example is a reformulation of a model considered by Phelps and Pollak (1968).

Substituting the first-order conditions for this utility maximization problem: $c_{\tau+1} = (\beta\delta(1+r))^{\frac{1}{\rho}} c_\tau$, and $c_{t+1} = (\delta(1+r))^{\frac{1}{\rho}} c_t$ for $t \geq \tau+1$, into the budget constraint (which holds as equality), we find that for the agent with quasi-hyperbolic discounting the optimal date-$\tau$ consumption in the date-$\tau$ problem, $c_\tau^*$, is a constant fraction of their expected total wealth:

$$\begin{aligned} c_\tau^* &= \frac{(1+r)s_{\tau-1}^* + \sum_{t=0}^{\infty} \frac{w}{(1+r)^t}}{1 + (\beta\delta)^{\frac{1}{\rho}}(1+r)^{\frac{1-\rho}{\rho}} + \ldots + (\beta\delta^t)^{\frac{1}{\rho}}(1+r)^{\frac{t(1-\rho)}{\rho}} + \ldots} \\ &= \frac{1 - \delta^{\frac{1}{\rho}}(1+r)^{\frac{1-\rho}{\rho}}}{1 - \delta^{\frac{1}{\rho}}(1+r)^{\frac{1-\rho}{\rho}} + (\beta\delta)^{\frac{1}{\rho}}(1+r)^{\frac{1-\rho}{\rho}}} \left( (1+r)s_{\tau-1}^* + \sum_{t=0}^{\infty} \frac{w}{(1+r)^t} \right). \end{aligned}$$

Respectively, the optimal date-$\tau$ savings are given by $s_\tau^* = (1+r)s_{\tau-1}^* + w - c_\tau^*$. Note that the propensity to consume out of the expected total wealth depends on $r$, but does not depend on $s_{\tau-1}^*$ and $w$. Thus, given $s_{-1}^*$, we can recursively construct a naive path $\{c_\tau^*, s_\tau^*\}_{\tau=0}^{\infty}$.

Let us compare this naive path with the solution to the similar problem for an agent with exponential discounting:

$$\max_{c_t \geq 0} \sum_{t=0}^{\infty} \gamma^t u(c_t), \qquad \text{s. t.} \sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t} \leq (1+r)s_{-1}^* + \sum_{t=0}^{\infty} \frac{w}{(1+r)^t}.$$

Repeating the same argument, it is easily seen that the solution to the utility maximization problem for the agent with $\gamma$ discounting, $\{c_\tau^{**}\}_{\tau=0}^{\infty}$, is recursively constructed by

$$c_\tau^{**} = \left(1 - \gamma^{\frac{1}{\rho}}(1+r)^{\frac{1-\rho}{\rho}}\right) \left( (1+r)s_{\tau-1}^{**} + \sum_{t=0}^{\infty} \frac{w}{(1+r)^t} \right),$$

where the sequence of savings $\{s_\tau^{**}\}_{\tau=0}^{\infty}$ is given by $s_\tau^{**} = (1+r)s_{\tau-1}^{**} + w - c_\tau^{**}$. Hence for the agent with exponential discounting the optimal date-$\tau$ consumption for any $\tau$ is also a constant fraction of expected total wealth.

Now consider the following question: given $\beta$ and $\delta$, can we find $\gamma$ such that $\{c_\tau^*, s_\tau^*\}_{\tau=0}^{\infty}$ coincides with $\{c_\tau^{**}, s_\tau^{**}\}_{\tau=0}^{\infty}$? This question is discussed under the header of "observational equivalence" (see Barro, 1999, for a significant contribution). It is easily seen that in this example the answer is affirmative: when $\gamma$ is chosen such that

$$\gamma^{\frac{1}{\rho}} \;=\; \frac{(\beta\delta)^{\frac{1}{\rho}}}{1-\delta^{\frac{1}{\rho}}(1+r)^{\frac{1-\rho}{\rho}}+(\beta\delta)^{\frac{1}{\rho}}(1+r)^{\frac{1-\rho}{\rho}}}\,,$$

for all $\tau$, $c_\tau^* = c_\tau^{**}$ and $s_\tau^* = s_\tau^{**}$, and thus $\{c_\tau^*, s_\tau^*\}_{\tau=0}^\infty$ and $\{c_\tau^{**}, s_\tau^{**}\}_{\tau=0}^\infty$ coincide.

To some extent the above argument can be generalized to the case where the interest and wage rates change over time. Suppose that utility is logarithmic, and the agent with quasi-hyperbolic discounting at date $\tau$ solves the problem:

$$\max_{c_t\geq 0} \;\; \ln c_\tau + \beta \sum_{t=\tau+1}^{\infty} \delta^{t-\tau} \ln c_t\,, \qquad \text{s. t. } \; c_\tau + \sum_{t=\tau+1}^{\infty} \frac{c_t}{(1+r_{\tau+1})\cdots(1+r_t)}$$
$$\leq\; (1+r_\tau)s_{\tau-1}^* + w_\tau + \sum_{t=\tau+1}^{\infty} \frac{w_t}{(1+r_{\tau+1})\cdots(1+r_t)}\,,$$

where $s_{\tau-1}^*$, $\{r_t\}_{t=\tau}^\infty$ and $\{w_t\}_{t=\tau}^\infty$ are taken as given by the agent.[11]

Again, substituting the first-order conditions: $c_{\tau+1} = \beta\delta(1+r_{\tau+1})c_\tau$, and $c_{t+1} = \delta(1+r_{t+1})c_t$ for $t \geq \tau+1$, into the budget constraint (which holds as equality), we obtain that for all $\tau$, the optimal date-$\tau$ consumption in the date-$\tau$ problem, $c_\tau^*$, for the agent with quasi-hyperbolic discounting is given by a constant fraction of expected total wealth:

$$c_\tau^* \;=\; \frac{1-\delta}{1-\delta+\beta\delta}\left((1+r_\tau)s_{\tau-1}^* + w_\tau + \sum_{t=\tau+1}^{\infty} \frac{w_t}{(1+r_{\tau+1})\cdots(1+r_t)}\right).$$

Similarly, for the agent with exponential discounting and log-utility, the optimal date-$\tau$ consumption, $c_\tau^{**}$, provided that initial savings are $s_{\tau-1}^*$, is given by

$$c_\tau^{**} = (1-\gamma)\left((1+r_\tau)s_{\tau-1}^* + w_\tau + \sum_{t=\tau+1}^{\infty} \frac{w_t}{(1+r_{\tau+1})\cdots(1+r_t)}\right).$$

It is easily seen that $c_\tau^* = c_\tau^{**}$ for all $\tau$ if and only if $\gamma = \frac{\beta\delta}{1-\delta+\beta\delta}$. Moreover, since a naive agent with $\beta$–$\delta$ discounting constantly revises their consumption path, the

[11]This example resembles the model considered by Barro (1999).

result of these revisions exactly coincides with the optimal consumption path for the agent with $\gamma$ discounting. Thus, there is again observational equivalence.

In the above two simple examples, we obtain observational equivalence from the perspective of a naive agent taking the interest and wage rates as given. It is natural to conjecture that these two examples can be extended to a more general framework that includes both the consumption and production sides of the economy. This argument suggests that observational equivalence is a fairly general phenomenon. However, our analysis shows that this is not the case.

In what follows we clarify the ideas about observational equivalence in a general equilibrium framework. We introduce sliding optimal paths (SOPs) and distinguish between two types of sliding equilibrium paths (SEPs): a SEP under pseudo-perfect foresight, which is a decentralization of a SOP, and a SEP under perfect foresight, which is a novel object. We study their properties and prove that, within the class of isoelastic utility functions, observational equivalence of SEPs under perfect foresight does not hold, except for the considered above cases of a constant interest rate and log-utility.

## 2.2 Summary of useful facts

To proceed further, it is necessary to recall the well-known facts and results from the discrete-time Ramsey model that will be referred to in the subsequent analysis.

Consider the **standard Ramsey model**, i.e., the optimal growth model with exponential discounting (Ramsey, 1928). Given an initial capital stock $k_0 > 0$, the planner solves the following problem at date 0:

$$\max_{c_t \geq 0,\ k_{t+1} \geq 0} \sum_{t=0}^{\infty} \gamma^t u(c_t)\,, \qquad \text{s. t. } \ c_t + k_{t+1} = f(k_t), \quad t \geq 0\,. \tag{1}$$

Here and in what follows we assume that capital depreciates completely within the period, and the production function $f(k)$ satisfies the standard assumptions: $f(0) = 0$, $f'(k) > 0$, $f''(k) < 0$, $\exists\ \bar{k} : f(\bar{k}) = \bar{k}$, and $\gamma f'(0) > 1$. We call a solution to problem (1) the **$\gamma$-optimal path** starting from $k_0$.

The optimal path in the standard Ramsey model can be decentralized as an equilibrium path. Consider the general equilibrium version of the standard Ramsey model (Cass, 1965; Koopmans, 1965). At the consumption side, a representative agent solves the following problem at date 0, given initial savings $s_{-1} = k_0$:

$$\max_{c_t \geq 0} \sum_{t=0}^{\infty} \gamma^t u(c_t)\,, \qquad \text{s. t. } c_0 + \sum_{t=1}^{\infty} \frac{c_t}{(1+r_1)\cdots(1+r_t)} \\ \leq (1+r_0)s_{-1} + w_0 + \sum_{t=1}^{\infty} \frac{w_t}{(1+r_1)\cdots(1+r_t)}\,, \tag{2}$$

where interest rates $\{r_t\}_{t=0}^{\infty}$ and wage rates $\{w_t\}_{t=0}^{\infty}$ are taken as given by the agent.

At the production side, a representative firm with production function $f(k)$ at each date $t$ takes as given the interest rate $r_t$ and solves the following myopic profit maximization problem, treating the residual profit as the wage rate:

$$\max_{k_t \geq 0} \; f(k_t) - (1+r_t)k_t. \tag{3}$$

The equilibrium path in the standard Ramsey model with the discount factor $\gamma$, which we call a **$\gamma$-equilibrium path** starting from $s_{-1}$, is a sequence $\{\tilde{c}_t, \tilde{s}_t, \tilde{k}_{t+1}, \tilde{r}_t, \tilde{w}_t\}_{t=0}^{\infty}$, defined as follows. First, the agent maximizes utility correctly foreseeing the prices, i.e., $\{\tilde{c}_t\}_{t=0}^{\infty}$ is the solution to problem (2) at given $\{\tilde{r}_t\}_{t=0}^{\infty}$ and $\{\tilde{w}_t\}_{t=0}^{\infty}$, and $\{\tilde{s}_t\}_{t=0}^{\infty}$ are determined by $\tilde{s}_t = (1+\tilde{r}_t)\tilde{s}_{t-1} + \tilde{w}_t - \tilde{c}_t$. Second, the firm maximizes profits, so prices are equal to marginal products: $1+\tilde{r}_t = f'(\tilde{k}_t)$ and $\tilde{w}_t = f(\tilde{k}_t) - f'(\tilde{k}_t)\tilde{k}_t$. Third, the capital market clears at each date, i.e., savings are equal to investment: $\tilde{s}_{t-1} = \tilde{k}_t$.

It is well known that in the standard Ramsey model *equilibrium and optimal paths are essentially the same* — the sequence $\{\tilde{c}_t, \tilde{k}_{t+1}\}_{t=0}^{\infty}$ extracted from the $\gamma$-equilibrium path starting from $s_{-1}$ solves problem (1), i.e., coincides with the $\gamma$-optimal path starting from $k_0 = s_{-1}$.

The solution to problem (2) is a consumer optimum at given interest and wage rates under exponential discounting. Under **quasi-hyperbolic discounting**, we define a date-$\tau$ consumer optimum similarly. Given initial savings $s_{\tau-1}$, consider the following problem at date $\tau$:

$$\max_{c_t \ge 0} \quad u(c_\tau) + \beta \sum_{t=\tau+1}^{\infty} \delta^{t-\tau} u(c_t)\,, \qquad \text{s. t. } \; c_\tau + \sum_{t=\tau+1}^{\infty} \frac{c_t}{(1+r_{\tau+1})\cdots(1+r_t)} \\ \le (1+r_\tau)s_{\tau-1} + w_\tau + \sum_{t=\tau+1}^{\infty} \frac{w_t}{(1+r_{\tau+1})\cdots(1+r_t)}\,, \tag{4}$$

where $\{r_t\}_{t=\tau}^{\infty}$ and $\{w_t\}_{t=\tau}^{\infty}$ are taken as given by the agent.

We call a sequence $\{c_t^\tau, s_t^\tau\}_{t=\tau}^{\infty}$, a **date-$\tau$ consumer optimum starting from $s_{\tau-1}$ at given $\{r_t\}_{t=\tau}^{\infty}$ and $\{w_t\}_{t=\tau}^{\infty}$**, if $\{c_t^\tau\}_{t=\tau}^{\infty}$ is a solution to problem (4) and $\{s_t^\tau\}_{t=\tau}^{\infty}$ are determined recursively by $s_t^\tau = (1+r_t)s_{t-1}^\tau + w_t - c_t^\tau$.

If a solution to problem (4) exists, it satisfies the budget constraint as equality and the following first-order conditions: $c_{\tau+1}^\tau = (\beta\delta(1+r_{\tau+1}))^{\frac{1}{\rho}} c_\tau^\tau$, and $c_t^\tau = (\delta(1+r_t))^{\frac{1}{\rho}} c_{t-1}^\tau$ for $t \ge \tau+2$. Substituting the first-order conditions into the budget constraint, we obtain the expression for the date-$\tau$ consumption in a date-$\tau$ consumer optimum:

$$c_\tau^\tau = \frac{(1+r_\tau)s_{\tau-1} + w_\tau + \sum_{t=\tau+1}^{\infty} \frac{w_t}{(1+r_{\tau+1})\cdots(1+r_t)}}{1 + (\beta\delta)^{\frac{1}{\rho}}(1+r_{\tau+1})^{\frac{1-\rho}{\rho}} + \ldots + (\beta\delta^t)^{\frac{1}{\rho}}\left((1+r_{\tau+1})\cdots(1+r_{\tau+t})\right)^{\frac{1-\rho}{\rho}} + \ldots}. \tag{5}$$

# 3 Sliding optimal paths

The main contribution of this paper is the study of a sliding equilibrium path under perfect foresight. To understand this notion, it is essential to distinguish between three different objects — sliding optimal path (SOP), sliding equilibrium path (SEP) under pseudo-perfect foresight, and SEP under perfect foresight.

To make our exposition more transparent, it is important to define a SOP and a SEP under pseudo-perfect foresight first. In this section we define a SOP, which is a natural concept to describe the behavior of a time-inconsistent planner. Further, we decentralize a SOP and note that on the corresponding equilibrium path an agent at each date revises their expectations about prices, which implies that their foresight is "pseudo-perfect". We introduce a SEP under pseudo-perfect foresight as a decentralized SOP, and characterize its properties.

## 3.1 Definition

Our main workhorse is the discrete-time Ramsey model with quasi-hyperbolic discounting. Given an initial capital stock $k_\tau > 0$, consider the following date-$\tau$ utility maximization problem:

$$\max_{c_t \geq 0,\ k_{t+1} \geq 0} \quad u(c_\tau) + \beta \sum_{t=\tau+1}^{\infty} \delta^{t-\tau} u(c_t)\,, \qquad \text{s. t.} \;\; c_t + k_{t+1} = f(k_t), \quad t \geq \tau\,. \quad (6)$$

A solution to problem (6), $\{c_t^{*\tau}, k_{t+1}^{*\tau}\}_{t=\tau}^{\infty}$, is the **date-$\tau$ optimal path** (under quasi-hyperbolic discounting) starting from $k_\tau$.

Note that the date-$\tau$ optimal path differs from the truncation of the optimal path at any previous date. Indeed, the discount factor between periods $\tau+1$ and $\tau$ equals $\beta\delta$ from the date $\tau$ perspective, while it is equal to $\delta$ from any earlier perspective. Therefore, the value $c_t^{*\tau}$ planned at date $\tau$ for the date-$t$ consumption will not be optimal when date $t$ comes, which implies that optimal paths under quasi-hyperbolic discounting are **time-inconsistent**.

We assume that the planner is naive and does not recognize their time inconsistency. A natural way to describe the behavior of a naive planner is to consider a step-by-step procedure where the planner at each date revises their optimal path and implements only the first step. We call the outcome of this procedure a sliding optimal path (SOP). Formally, the following definition applies.

**Definition 1.** *A sequence $\{c_t^\circ, k_{t+1}^\circ\}_{t=0}^{\infty}$ is a **sliding optimal path** starting from $k_0$ in the Ramsey model with quasi-hyperbolic discounting, if for each $\tau \geq 0$, consumption and capital stock at date $\tau$ are obtained from the date-$\tau$ optimal path under quasi-hyperbolic discounting starting from $k_\tau^\circ$: $c_\tau^\circ = c_\tau^{*\tau}$ and $k_{\tau+1}^\circ = k_{\tau+1}^{*\tau}$.*

Clearly, *a SOP exists and is unique.* Note that a SOP is essentially characterized by the first step in problem (6), i.e., by the first elements from the date-$\tau$ optimal path $\{c_t^{*\tau}, k_{t+1}^{*\tau}\}_{t=\tau}^{\infty}$. After the first step is implemented at date $\tau$ ($c_\tau^{*\tau}$ is consumed and $k_{\tau+1}^{*\tau}$ remains as the new capital stock), the planner in fact solves problem (1) with the constant discount factor $\delta$. Therefore, the truncation of the date-$\tau$ optimal path which starts at date $\tau+1$, $\{c_t^{*\tau}, k_{t+1}^{*\tau}\}_{t=\tau+1}^{\infty}$, is the $\delta$-optimal path starting from $k^{*\tau}_{\tau+1}$. It follows that $c^{*\tau}_t$ and $k^{*\tau}_{t+1}$ converge to the corresponding modified golden rule consumption and capital stock for the discount factor $\delta$.

This observation allows us to describe the first step in problem (6) and therefore a SOP in terms of dynamic programming (see Appendix A). The interesting question to ask is whether a SOP under quasi-hyperbolic discounting coincides with some $\gamma$-optimal path. We formally define observational equivalence of optimal paths as follows.

**Definition 2.** *A sliding optimal path in the Ramsey model with quasi-hyperbolic discounting, $\{c^\circ_t, k^\circ_{t+1}\}_{t=0}^\infty$, is* ***observationally equivalent*** *to the $\gamma$-optimal path if there exists $\gamma$ for which $\{c^\circ_t, k^\circ_{t+1}\}_{t=0}^\infty$ is a solution to problem* (1).

To the best of our knowledge, a SOP is observationally equivalent to a $\gamma$-optimal path only in the following two cases. First, in the case of a stationary sliding optimum (SSO).

**Definition 3.** *A pair $\{c^\circ, k^\circ\}$ is a* ***stationary sliding optimum*** *if the sequence $\{c^\circ_t, k^\circ_{t+1}\}_{t=0}^\infty$, where for each $t \geq 0$, $c^\circ_t = c^\circ$ and $k^\circ_{t+1} = k^\circ$, is a sliding optimal path starting from $k^\circ$.*

Clearly, a SSO is observationally equivalent to a stationary $\gamma^\circ$-optimum for $\gamma^\circ = 1/f'(k^\circ)$. It can be checked that $\gamma^\circ < \delta$ (see Appendix A).

Second, in the case of log-utility and Cobb–Douglas production technology.

**Claim 1.** *Suppose that $u(c) = \ln c$ and $f(k) = k^\alpha$. Then a sliding optimal path in the Ramsey model with quasi-hyperbolic discounting is observationally equivalent to the $\gamma^\circ$-optimal path, where $\gamma^\circ = \frac{\beta\delta}{1-\alpha\delta+\alpha\beta\delta}$.*

*Proof.* See Appendix A. ■

Claim 1 implies that in the simple case of log-utility and Cobb–Douglas production technology, observing only the path $\{c^\circ_t, k^\circ_{t+1}\}_{t=0}^\infty$, one cannot determine whether the planner has $\beta$–$\delta$ discounting and is time-inconsistent; or the planner has $\gamma^\circ$ discounting and is time-consistent. The equivalent discount factor $\gamma^\circ$ lies in between the short-run discount factor $\beta\delta$ and the long-run discount factor $\delta$ ($\beta\delta < \gamma^\circ < \delta$), and depends on the technology parameter $\alpha$.

## 3.2 Decentralization of sliding optimal paths

The natural question is whether a SOP can be decentralized as an equilibrium path. Since a SOP is constructed by applying at each date $\tau$ the corresponding date-$\tau$ optimal path, to decentralize a SOP we need to decentralize a date-$\tau$ optimal path as a date-$\tau$ equilibrium path.

Consider the date-$\tau$ general equilibrium version of the Ramsey model with quasi-hyperbolic discounting. A representative agent solves problem (4). A representative firm at each date $t \geq \tau$ solves problem (3).

The definition of equilibrium is similar to that of the standard Ramsey model (see Section 2.2). A **date-$\tau$ equilibrium path** (under quasi-hyperbolic discounting) starting from $s_{\tau-1}$ is a sequence $\{c_t^{*\tau}, s_t^{*\tau}, k_{t+1}^{*\tau}, r_t^{*\tau}, w_t^{*\tau}\}_{t=\tau}^{\infty}$, such that i) the agent maximizes utility correctly foreseeing the prices from the date $\tau$ perspective: $\{c_t^{*\tau}, s_t^{*\tau}\}_{t=\tau}^{\infty}$ is a date-$\tau$ consumer optimum starting from $s_{\tau-1}$ at given $\{r_t^{*\tau}\}_{t=\tau}^{\infty}$ and $\{w_t^{*\tau}\}_{t=\tau}^{\infty}$; ii) at each date prices are equal to marginal products from the date $\tau$ perspective: $1 + r_t^{*\tau} = f'(k_t^{*\tau})$ and $w_t^{*\tau} = f(k_t^{*\tau}) - f'(k_t^{*\tau})k_t^{*\tau}$; iii) at each date savings are equal to investment: $s_{t-1}^{*\tau} = k_t^{*\tau}$.

Note that the *date-$\tau$ equilibrium and the date-$\tau$ optimal paths are essentially the same* — a sequence $\{c_t^{*\tau}, k_{t+1}^{*\tau}\}_{t=\tau}^{\infty}$ extracted from the date-$\tau$ equilibrium path is the date-$\tau$ optimal path (starting from $k_\tau = s_{\tau-1}$), i.e., the solution to problem (6). Clearly, the date-$\tau$ equilibrium path differs from the truncation of any previous-date equilibrium path, and hence at each date $\tau$ there arises a new equilibrium.

Thus, decentralization of a SOP is a sliding equilibrium path (SEP) obtained by constructing at each date $\tau$ the corresponding date-$\tau$ equilibrium path. It should be emphasized that, due to naivety, agent's expectations about prices are correct only from the date-$\tau$ perspective. Therefore, the agent revises their expectations at each date: on the date-$\tau$ equilibrium path, the date-$\tau$ consumer optimum is obtained under expectations $\{r_t^{*\tau}\}_{t=\tau}^{\infty}$ and $\{w_t^{*\tau}\}_{t=\tau}^{\infty}$, while on the date-$\tau'$ equilibrium path, date-$\tau'$ consumer optimum is obtained under different expectations $\{r_t^{*\tau'}\}_{t=\tau'}^{\infty}$ and $\{w_t^{*\tau'}\}_{t=\tau'}^{\infty}$. It turns out that the agent correctly foresees prices on the date-$\tau$ equilibrium path, but cannot correctly foresee prices on the SEP.

The agent has only "pseudo-perfect" foresight in this case, which motivates the following definition.

**Definition 4.** *A sequence* $\{c_t^\circ, s_t^\circ, k_{t+1}^\circ, r_t^\circ, w_t^\circ\}_{t=0}^\infty$ *is a* ***sliding equilibrium path under pseudo-perfect foresight*** *starting from* $s_{-1}^\circ$*, if*

1. *Consumption and savings at each date* $\tau$ *are obtained from the date-*$\tau$ *equilibrium path, i.e., from the date-*$\tau$ *consumer optimum starting from* $s_{\tau-1}^\circ$ *at given* $\{r_t^{*\tau}\}_{t=\tau}^\infty$ *and* $\{w_t^{*\tau}\}_{t=\tau}^\infty$*;*
2. *Prices at each date* $\tau$ *are equal to marginal products:* $1 + r_\tau^\circ = f'(k_\tau^\circ)$ *and* $w_\tau^\circ = f(k_\tau^\circ) - f'(k_\tau^\circ)k_\tau^\circ$*;*
3. *Savings at each date* $\tau$ *are equal to investment:* $s_\tau^\circ = k_{\tau+1}^\circ$*.*

The structure of a SEP under pseudo-perfect foresight is illustrated in Fig. 1.

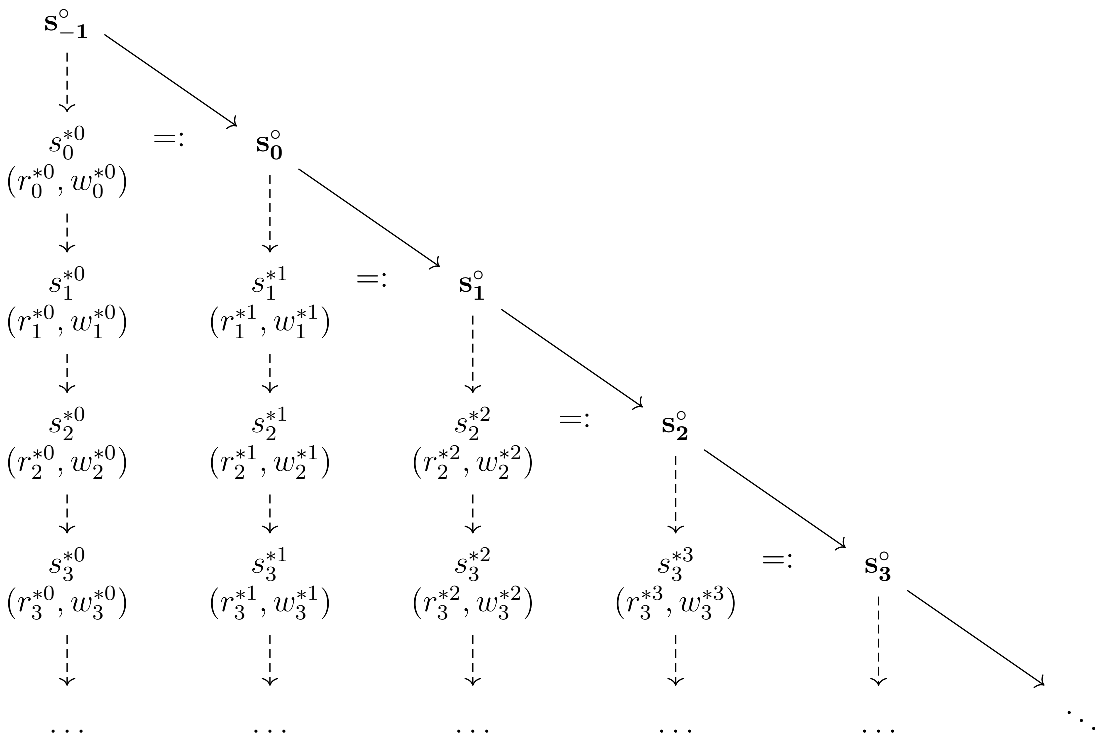


Figure 1: Savings on date-$\tau$ equilibria and SEP under pseudo-perfect foresight

Fig. 1 highlights the fact that a SEP under pseudo-perfect foresight depends on all date-$\tau$ equilibrium paths for $\tau \geq 0$, and suggests the following interpretation.

On a SEP under pseudo-perfect foresight, an agent at each date is naive about their time-inconsistent preferences (hence revising their consumer optimum), and also cannot perfectly foresee prices on a SEP (hence revising the expected prices). The agent correctly foresees date-$\tau$ equilibrium prices at each date $\tau$ (in each column in Fig. 1), but cannot take into account that the equilibrium itself will change. Hence the label **pseudo-perfect foresight**, which indicates the naivety of the agent about their expectations.

By construction, a SOP is decentralized as a SEP under pseudo-perfect foresight: a sequence $\{c_t^\circ, k_{t+1}^\circ\}_{t=0}^\infty$ extracted from a SEP under pseudo-perfect foresight is a SOP starting from $k_0 = s_{-1}$. It immediately follows that *a SEP under pseudo-perfect foresight exists and is unique.*

Clearly, the results about observational equivalence of SEPs under pseudo-perfect foresight are the same as that of SOPs. Formally, *a SEP under pseudo-perfect foresight is observationally equivalent to a $\gamma$-optimal path when the corresponding SOP is observationally equivalent to a $\gamma$-optimal path.* Therefore, for SEPs under pseudo-perfect foresight, observational equivalence holds only in two cases: the stationary case and the case of log-utility and Cobb–Douglas production technology (cf. Claim 1).

Consider the stationary case in more detail. Since a SEP under pseudo-perfect foresight is essentially the same as a SOP, the definition of a stationary sliding equilibrium (SSE) under pseudo-perfect foresight is straightforward.

**Definition 5.** *A tuple $\{c^\circ, s^\circ, k^\circ, r^\circ, w^\circ\}$ is a **stationary sliding equilibrium under pseudo-perfect foresight** if the sequence $\{c_t^\circ, s_t^\circ, k_{t+1}^\circ, r_t^\circ, w_t^\circ\}_{t=0}^\infty$, satisfying for each $t \geq 0$, $c_t^\circ = c^\circ$, $s_t^\circ = s^\circ$, $k_{t+1}^\circ = k^\circ$, $r_t^\circ = r^\circ$, and $w_t^\circ = w^\circ$, is a sliding equilibrium path under pseudo-perfect foresight starting from $k^\circ = s^\circ$.*

Recall that by the definition of SEP under pseudo-perfect foresight, consumption, savings, capital stock and prices, $\{c^\circ, s^\circ, k^\circ, r^\circ, w^\circ\}$, are the elements of the associated date-$\tau$ equilibrium path starting from $k^\circ$. It is clear that this underlying date-$\tau$ equilibrium path, $\{c_t^{*\tau}, s_t^{*\tau}, k_{t+1}^{*\tau}, r_t^{*\tau}, w_t^{*\tau}\}_{t=\tau}^\infty$, which determines a SSE under pseudo-perfect foresight, does not depend on $\tau$ (all columns in Fig. 1 are identical). It is characterized as follows: $\{c_\tau^{*\tau}, s_\tau^{*\tau}, k_{\tau+1}^{*\tau}, r_\tau^{*\tau}, w_\tau^{*\tau}\} = \{c^\circ, s^\circ, k^\circ, r^\circ, w^\circ\}$, and $\{c_t^{*\tau}, s_t^{*\tau}, k_{t+1}^{*\tau}, r_t^{*\tau}, w_t^{*\tau}\}_{t=\tau+1}^\infty$ is a $\delta$-equilibrium path starting from $k^\circ = s^\circ$. Note that the sequence of interest rates on this date-$\tau$ equilibrium path is not constant and has the form $\left\{r^\circ, r^\circ, \{r_t^{*\tau}\}_{t=\tau+2}^\infty\right\}$, where $\{r_t^{*\tau}\}_{t=\tau+2}^\infty$ decreases and converges to the modified golden rule interest rate for the discount factor $\delta$.

# 4 Sliding equilibrium path under perfect foresight

Now we turn to the central object of our study, a sliding equilibrium path under perfect foresight. Section 4.1 provides a formal definition of a SEP under perfect foresight. In Sections 4.2 and 4.3 we analyze the log-utility case and the stationary case respectively. Section 4.4 reports the general results concerning the existence of a SEP under perfect foresight, and its observational equivalence. Section 4.5 compares SEPs under pseudo-perfect and perfect foresight.

## 4.1 Definition

We introduce perfect foresight, as opposed to pseudo-perfect foresight. The agent with perfect foresight is only partially naive, correctly foreseeing prices on a SEP, but remaining unaware of their time inconsistency and revising consumer optimum. At each date, the agent cannot resist the temptation to consume more than exponential discounting would prescribe. This leads to the following definition.

**Definition 6.** *A sequence $\{c_t^*, s_t^*, k_{t+1}^*, r_t^*, w_t^*\}_{t=0}^\infty$ is a* ***sliding equilibrium path under perfect foresight*** *starting from $s_{-1}^*$, if*

1. *Consumption and savings at each date $\tau$ are obtained from the date-$\tau$ consumer optimum starting from $s_{\tau-1}^*$ at given $\{r_t^*\}_{t=\tau}^\infty$ and $\{w_t^*\}_{t=\tau}^\infty$;*
2. *Prices at each date $\tau$ are equal to marginal products: $1 + r_\tau^* = f'(k_\tau^*)$ and $w_\tau^* = f(k_\tau^*) - f'(k_\tau^*)k_\tau^*$;*
3. *Savings at each date $\tau$ are equal to investment: $s_\tau^* = k_{\tau+1}^*$.*

Thus, a SEP under perfect foresight is associated with an infinite sequence of corresponding consumer optima, i.e., an infinite sequence of optimization problems of form (4). Indeed, a SEP under perfect foresight is a sequence $\{c_t^*, s_t^*, k_{t+1}^*, r_t^*, w_t^*\}_{t=0}^{\infty}$ which is characterized as follows:

- there is a date-0 consumer optimum starting from $s_{-1}^*$ at given $\{r_t^*\}_{t=0}^{\infty}$ and $\{w_t^*\}_{t=0}^{\infty}$, which we denote by $\{c_t^{**0}, s_t^{**0}\}_{t=0}^{\infty}$, and its first elements are precisely the date-0 consumption and savings on a SEP under perfect foresight: $c_0^{**0} = c_0^*$ and $s_0^{**0} = s_0^*$;
- there is a date-1 consumer optimum starting from $s_0^*$ at given $\{r_t^*\}_{t=1}^{\infty}$ and $\{w_t^*\}_{t=1}^{\infty}$ (truncated sequences of sliding equilibrium prices), denoted by $\{c_t^{**1}, s_t^{**1}\}_{t=1}^{\infty}$, and its first elements are the date-1 consumption and savings on a SEP under perfect foresight: $c_1^{**1} = c_1^*$ and $s_1^{**1} = s_1^*$;
- and so forth; so that the resulting capital stock sequence $k_{t+1}^* = s_t^* = s_t^{**t}$ determines the sliding equilibrium sequences of interest and wage rates which are correctly expected by the agent solving for consumer optima at each date.

The construction of a SEP under perfect foresight is illustrated in Fig. 2.

By comparing Definitions 4 and 6, as well as Fig. 1 and 2, it can be seen that the important difference between SEPs under pseudo-perfect and perfect foresight is the formation of price expectations in the consumer optimum. Under pseudo-perfect foresight, the sequences of interest and wage rates expected at date $\tau$ coincide with those realized on the date-$\tau$ equilibrium path, but not on the SEP. Under perfect foresight, the sequences of interest and wage rates expected at date $\tau$ coincide with those realized on the SEP, and hence they are the same for different dates $\tau$ (i.e., in each row in Fig. 2).

Again, of particular interest is the question of observational equivalence of SEPs under perfect foresight.

**Definition 7.** *A sliding equilibrium path under perfect foresight in the Ramsey model with quasi-hyperbolic discounting starting from* $s_{-1}^*$, $\{c_t^*, s_t^*, k_{t+1}^*, r_t^*, w_t^*\}_{t=0}^{\infty}$,

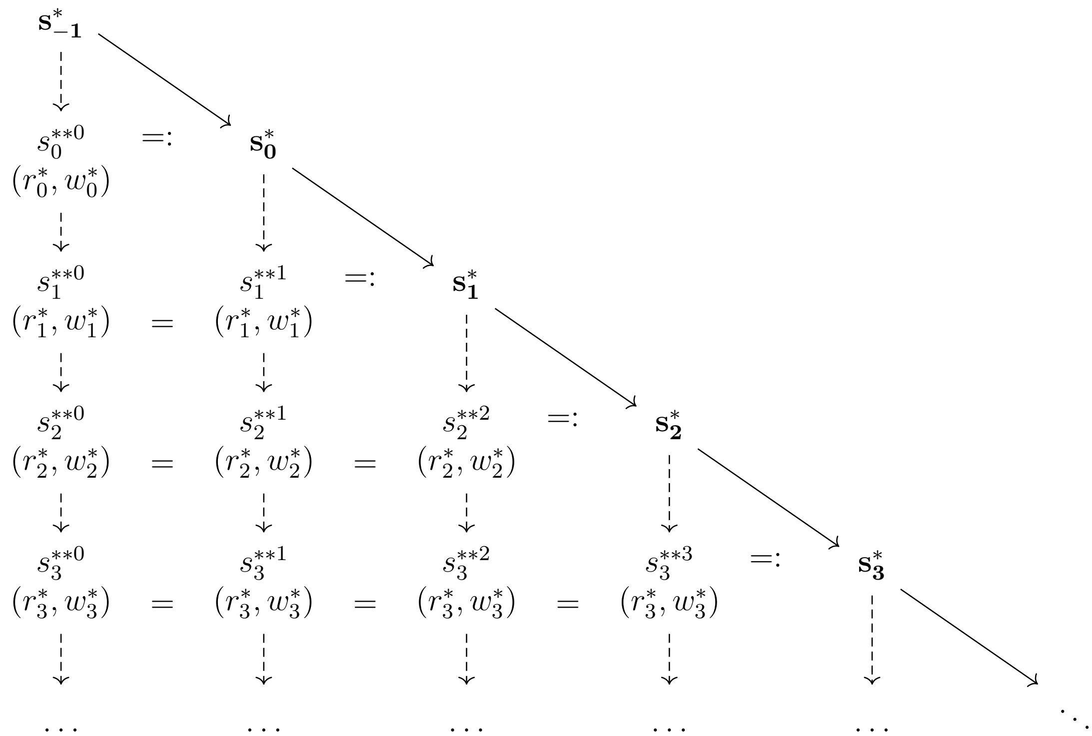


Figure 2: Savings on consumer optima and a SEP under perfect foresight

*is* ***observationally equivalent*** *to a $\gamma$-optimal path, if there exists $\gamma$ for which the sequence $\{c_t^*, k_{t+1}^*\}_{t=0}^\infty$ is a solution to problem* (1) *starting from $k_0^* = s_{-1}^*$.*

## 4.2 Logarithmic utility

It should be emphasized that Definition 6 of a SEP under perfect foresight and Definition 7 of observational equivalence clarify the ideas of Barro (1999). To make this relation and our definitions more transparent, let us consider the particular yet important log-utility case. As in Barro (1999), the following result holds.

**Claim 2.** *Suppose that $u(c) = \ln c$. Then, irrespective of the production technology, a sliding equilibrium path under perfect foresight exists, is unique, and is observationally equivalent to the $\gamma^*$-optimal path, where $\gamma^* = \frac{\beta\delta}{1-\delta+\beta\delta}$.*

*Proof.* It follows from (5) that when $\rho = 1$, the date-$\tau$ consumption in the date-$\tau$ consumer optimum starting from $s_{\tau-1}^*$ at given $\{r_t^*\}_{t=\tau}^\infty$ and $\{w_t^*\}_{t=\tau}^\infty$, satisfies $c_\tau^{**\tau} = \frac{1-\delta}{1-\delta+\beta\delta} M_\tau^*$, where $M_\tau^* = (1 + r_\tau^*)s_{\tau-1}^* + w_\tau^* + \sum_{t=\tau+1}^\infty \frac{w_t^*}{(1+r_{\tau+1}^*)\cdots(1+r_t^*)}$ is the present (date-$\tau$) value of the expected date-$\tau$ lifetime income. For the formal argument that $M_\tau^* < \infty$, see the proof of Lemma 2.3 in Appendix C.

By definition, a SEP under perfect foresight starting from $s_{-1}^*$ is a sequence $\{c_t^*, s_t^*, k_{t+1}^*, r_t^*, w_t^*\}_{t=0}^{\infty}$ such that for all $\tau \geq 0$,

$$c_\tau^* = \frac{1-\delta}{1-\delta+\beta\delta} M_\tau^*, \qquad k_{\tau+1}^* = s_\tau^* = (1+r_\tau^*)s_{\tau-1}^* + w_\tau^* - c_\tau^*,$$
$$1 + r_\tau^* = f'(k_\tau^*), \qquad w_\tau^* = f(k_\tau^*) - f'(k_\tau^*)k_\tau^*.$$

It follows that lifetime incomes expected at dates $\tau$ and $\tau + 1$ are linked: $M_\tau^* - (1 + r_\tau^*)s_{\tau-1}^* - w_\tau^* = \frac{M_{\tau+1}^*}{1+r_{\tau+1}^*} - s_\tau^*$, and hence $M_{\tau+1}^* = (1 + r_{\tau+1}^*)(M_\tau^* - c_\tau^*)$. Therefore,

$$c_{\tau+1}^* = \frac{1-\delta}{1-\delta+\beta\delta} M_{\tau+1}^* = (1+r_{\tau+1}^*)\left(c_\tau^* - \frac{1-\delta}{1-\delta+\beta\delta} c_\tau^*\right) = \frac{\beta\delta}{1-\delta+\beta\delta}(1+r_{\tau+1}^*)c_\tau^*.$$

Thus on a SEP under perfect foresight, the consumption levels at two adjacent dates are linked via the following “first-order conditions”: $c_{t+1}^* = \frac{\beta\delta}{1-\delta+\beta\delta}(1+r_{t+1}^*)c_t^*$. Now it is clear that the sequence $\{c_t^*, k_{t+1}^*\}_{t=0}^{\infty}$ extracted from a SEP under perfect foresight is the $\gamma^*$-optimal path, where $\gamma^* = \frac{\beta\delta}{1-\delta+\beta\delta}$, and hence observational equivalence holds. ■

Three comments about Claim 2 are in order. First, Claim 2 can be compared to the results of Barro (1999). His argument implies that the agent revises their consumption path at each instant of time, but does not revise their expectations. Hence Barro implicitly assumes a naive agent who correctly foresees prices taking into account that the equilibrium will change. Thus Barro (1999) considers a SEP under perfect foresight in terms of our Definition 6, and our result about observational equivalence under log-utility naturally confirms his findings.

Second, Krusell et al. (2002) prove observational equivalence of a sophisticated path in the Ramsey model with quasi-hyperbolic discounting, log-utility and Cobb–Douglas production technology, and obtain the same formula for the equivalent discount factor $\gamma^*$. However, Krusell et al. (2002) consider a sophisticated agent, while we consider a naive agent; and they consider only a Cobb–Douglas technology, while our result holds irrespective of the production technology (and $\gamma^*$ does not depend on technology).

Third, Claim 2 establishes the link between observational equivalence and a controlled comparison of discount functions (see, e.g., Myerson et al., 2001; Caliendo and Findley, 2014). Note that the equivalent discount factor is such that $\gamma^*$ discounting provides the same degree of overall impatience as $\beta$–$\delta$ discounting: $\sum_{t=0}^{\infty}(\gamma^*)^t = 1 + \beta\sum_{t=1}^{\infty}\delta^t$. Under log-utility, controlling for overall impatience implies that the paths of consumption and capital under exponential and quasi-hyperbolic discounting are observationally equivalent.

## 4.3 Stationary sliding equilibrium

Before turning to the general results about transitional paths, let us consider a stationary sliding equilibrium (SSE) under perfect foresight.

**Definition 8.** *A tuple $\{c^*, s^*, k^*, r^*, w^*\}$ is a* ***stationary sliding equilibrium under perfect foresight*** *if the sequence $\{c_t^*, s_t^*, k_{t+1}^*, r_t^*, w_t^*\}_{t=0}^{\infty}$, satisfying for each $t \geq 0$, $c_t^* = c^*$, $s_t^* = s^*$, $k_{t+1}^* = k^*$, $r_t^* = r^*$, and $w_t^* = w^*$, is a sliding equilibrium path under perfect foresight starting from $s^*$.*

By the definition of SEP under perfect foresight, $\{c^*, s^*\}$ are obtained from the associated date-$\tau$ consumer optimum starting from $s^*$ at given constant interest rate $r^*$ and wage rate $w^*$. It is clear that this underlying date-$\tau$ consumer optimum, which determines a SSE under perfect foresight, does not depend on $\tau$.

An important difference between SSEs under pseudo-perfect and perfect foresight lies in the formation of price expectations in the consumer optimum. As we have seen, consumption and savings on a SSE under pseudo-perfect foresight, $\{c^\circ, s^\circ\}$, are the first elements of the consumer optimum starting from $s^\circ$ at given $\left\{r^\circ, r^\circ, \{r_t^{*\tau}\}_{t=\tau+2}^{\infty}\right\}$ and $\left\{w^\circ, w^\circ, \{w_t^{*\tau}\}_{t=\tau+2}^{\infty}\right\}$. At the same time, consumption and savings on a SSE under perfect foresight, $\{c^*, s^*\}$, are the first elements of the consumer optimum starting from $s^*$ at given $\{r^*, r^*, r^*, \ldots\}$ and $\{w^*, w^*, w^*, \ldots\}$.

The following theorem maintains that a SSE under perfect foresight exists, is unique and there is always observational equivalence.

**Theorem 1.** *There is a unique* stationary *sliding equilibrium under perfect foresight. It is observationally equivalent to a* stationary $\gamma^*$*-optimum, where*

$$\gamma^* = \frac{(\gamma^*)^{\frac{1}{\rho}} - (\beta\delta)^{\frac{1}{\rho}}}{\delta^{\frac{1}{\rho}} - (\beta\delta)^{\frac{1}{\rho}}}. \tag{7}$$

*Proof.* Take $r^* > 0$ and some $w^* > 0$. Let $\{c_t^{**\tau}\}_{t=\tau}^{\infty}$ be consumption in a date-$\tau$ consumer optimum starting from $s^*$ at given constant interest rate $r^*$ and wage rate $w^*$, i.e., the solution to problem (4) for $\{r_t\}_{t=\tau}^{\infty} = \{r^*, r^*, \ldots\}$ and $\{w_t\}_{t=\tau}^{\infty} = \{w^*, w^*, \ldots\}$. The following lemma characterizes this consumer optimum.

**Lemma 1.1.** *Suppose that $\delta(1+r^*)^{1-\rho} < 1$. A date-$\tau$ consumer optimum starting from $s^*$ at given $r^*$ and $w^*$ exists, is unique, and the date-$\tau$ consumption satisfies*

$$c_\tau^{**\tau} = \frac{1 - \delta^{\frac{1}{\rho}}(1+r^*)^{\frac{1-\rho}{\rho}}}{1 - \delta^{\frac{1}{\rho}}(1+r^*)^{\frac{1-\rho}{\rho}} + (\beta\delta)^{\frac{1}{\rho}}(1+r^*)^{\frac{1-\rho}{\rho}}} \cdot \frac{1+r^*}{r^*} \cdot (r^* s^* + w^*) \,. \tag{8}$$

*Proof.* See Appendix B. □

By the definition of SSE under perfect foresight, $c_\tau^{**\tau} = c^* = (1+r^*)s^* + w^* - s^* = r^* s^* + w^*$. It now follows from (8) that the interest rate in a SSE under perfect foresight is such that

$$1 + r^* = \frac{1 - \delta^{\frac{1}{\rho}}(1+r^*)^{\frac{1-\rho}{\rho}} + (\beta\delta)^{\frac{1}{\rho}}(1+r^*)^{\frac{1-\rho}{\rho}}}{(\beta\delta)^{\frac{1}{\rho}}(1+r^*)^{\frac{1-\rho}{\rho}}} \,,$$

i.e., $r^*$ satisfies

$$\frac{1}{1+r^*} = \frac{\left(\frac{1}{1+r^*}\right)^{\frac{1}{\rho}} - (\beta\delta)^{\frac{1}{\rho}}}{\delta^{\frac{1}{\rho}} - (\beta\delta)^{\frac{1}{\rho}}} \,. \tag{9}$$

The following lemma maintains that there exists a unique solution to equation (9), and it is compatible with the existence of a date-$\tau$ consumer optimum.

**Lemma 1.2.** *There is a unique solution $r^*$ to equation* (9)*, and $\delta(1+r^*)^{1-\rho} < 1$.*

*Proof.* See Appendix B. □

It follows that the tuple $\{c^*, s^*, k^*, r^*, w^*\}$ where $r^*$ is the solution to equation (9), and $s^* = k^* = (f')^{-1}(1 + r^*)$, $w^* = f(k^*) - f'(k^*)k^*$, and $c^* = r^* s^* + w^*$, is the unique SSE under perfect foresight.

Furthermore, denote $\gamma^* = \frac{1}{1+r^*}$. It is clear that since $r^*$ satisfies (9), $\gamma^*$ satisfies (7). Since $f'(k^*) = 1/\gamma^*$, it follows that $\{c^*, k^*\}$ is a stationary optimum in the standard Ramsey model with the discount factor $\gamma^*$. Therefore, a SSE under perfect foresight is observationally equivalent to a stationary $\gamma^*$-optimum. ■

It can be easily checked that the stationary equivalent discount factor $\gamma^*$ lies in between the short-run discount factor $\beta\delta$ and the long-run discount factor $\delta$ ($\beta\delta < \gamma^* < \delta$), and is increasing both in $\beta$ and in $\delta$. Note also that in the stationary case with log-utility, Theorem 1 and Claim 2 yield the same result.

## 4.4 General results

The following theorem proves the existence of a SEP under perfect foresight.

**Theorem 2.** *There exists a sliding equilibrium path under perfect foresight starting from any* $s^*_{-1} > 0$.

*Proof.* We prove the existence of a SEP under perfect foresight in two steps. First, we consider a SEP under perfect foresight in the finite horizon model and show that for any $T \in \mathbb{N}$ there exists a finite $T$-horizon SEP. Second, we construct a candidate for a SEP in the infinite horizon model by applying a diagonalization procedure to the sequence of finite $T$-horizon SEPs, and show that this candidate is indeed a SEP under perfect foresight.

Fix a finite horizon $T > 0$. For any date $0 \le \tau \le T$, consider the following $T$-horizon date-$\tau$ problem:

$$\max_{c_t \ge 0} \; u(c_\tau) + \beta \sum_{t=\tau+1}^{T+1} \delta^{t-\tau} u(c_t), \qquad \text{s. t. } c_\tau + \sum_{t=\tau+1}^{T+1} \frac{c_t}{(1 + r_{\tau+1}) \cdots (1 + r_t)} \le f(k_\tau) + \sum_{t=\tau+1}^{T+1} \frac{w_t}{(1 + r_{\tau+1}) \cdots (1 + r_t)}, \tag{10}$$

where the initial capital stock $k_\tau > 0$ and sequences of interest rates $\{r_{t+1}\}_{t=\tau}^{T}$ and wage rates $\{w_{t+1}\}_{t=\tau}^{T}$ are taken as given by the agent.

Similarly to the infinite horizon case (cf. Section 2.2), we call the sequence $\{c_t^\tau, s_t^\tau\}_{t=\tau}^{T+1}$ a **$T$-horizon date-$\tau$ consumer optimum starting from $k_\tau$ at given $\{r_{t+1}\}_{t=\tau}^{T}$ and $\{w_{t+1}\}_{t=\tau}^{T}$** if $\{c_t^\tau\}_{t=\tau}^{T+1}$ is the solution to problem (10), and $\{s_t^\tau\}_{t=\tau}^{T+1}$ is given recursively by $s_\tau^\tau = f(k_\tau) - c_\tau^\tau$ and $s_t^\tau = (1+r_t)s_{t-1}^\tau + w_t - c_t^\tau$.

A finite $T$-horizon SEP under perfect foresight is formally defined as follows.

**Definition 9.** *A sequence $\{c_t^*(T), s_t^*(T), k_{t+1}^*(T), r_{t+1}^*(T), w_{t+1}^*(T)\}_{t=0}^{T}$ is a **$T$-horizon sliding equilibrium path under perfect foresight** starting from $s_{-1}^* = k_0^*$ if*

1. *Consumption and savings at date $\tau$ are the elements of the $T$-horizon date-$\tau$ consumer optimum starting from $k_\tau^*(T)$ at given $\{r_{t+1}^*(T)\}_{t=\tau}^{T}$ and $\{w_{t+1}^*(T)\}_{t=\tau}^{T}$;*

2. *Prices at each date are equal to marginal products: for $1 \le t \le T+1$, $1 + r_t^*(T) = f'(k_t^*(T))$ and $w_t^*(T) = f(k_t^*(T)) - f'(k_t^*(T))k_t^*(T)$;*

3. *Savings at each date are equal to investment: $s_t^*(T) = k_{t+1}^*(T)$ for $0 \le t \le T$.*

**Lemma 2.1.** *There exists a $T$-horizon sliding equilibrium path under perfect foresight starting from any $s_{-1}^* = k_0^* > 0$.*

*Proof.* See Appendix C. □

Importantly enough, the sequence of capital stocks on a $T$-horizon SEP under perfect foresight is bounded from both below and above. To define the bounds, let $\{\underline{c}_t(T), \underline{k}_{t+1}(T)\}_{t=0}^{T+1}$ be a solution to the following problem given $\underline{k}_0 = k_0^*$:

$$\max_{c_t \ge 0,\ k_{t+1} \ge 0} \sum_{t=0}^{T+1} (\beta\delta)^t u(c_t)\,, \qquad \text{s. t. } \ c_t + k_{t+1} = f(k_t)\,, \quad 0 \le t \le T+1\,, \quad (11)$$

and let $\{\bar{c}_t(T), \bar{k}_{t+1}(T)\}_{t=0}^{T+1}$ be a solution to the following problem given $\bar{k}_0 = k_0^*$:

$$\max_{c_t \ge,\ k_{t+1} \ge 0} \sum_{t=0}^{T+1} \delta^t u(c_t)\,, \qquad \text{s. t. } \ c_t + k_{t+1} = f(k_t)\,, \quad 0 \le t \le T+1\,. \qquad (12)$$

**Lemma 2.2.** *For all $T \geq 1$, and for all $0 \leq t \leq T$,*

$$\underline{k}_{t+1}(T) \;<\; k^*_{t+1}(T) \;<\; \bar{k}_{t+1}(T)\,. \tag{13}$$

*Proof.* See Appendix C. □

Now consider the sequence $\left[\{k^*_{t+1}(T)\}_{t=0}^{T}\right]_{T\geq 1}$, whose elements are the sequences of capital stocks on the $T$-horizon SEPs under perfect foresight starting from the same $s^*_{-1} = k^*_0$, for increasing horizons $T = 1, 2, \ldots$.

Let us apply the following procedure to the sequence $\left[\{k^*_{t+1}(T)\}_{t=0}^{T}\right]_{T\geq 1}$. At the first step of the procedure, consider the sequence $\{k^*_1(T)\}_{T\geq 1}$, take a cluster point $k^*_1$ of this sequence and extract a subsequence $\{T_{1n}\}_{n=1}^{\infty}$ from $\{T\}_{T\geq 1}$ such that $\{k^*_1(T_{1n})\}_{n=1}^{\infty}$ converges to $k^*_1$. At the second step, consider the sequence $\{k^*_2(T_{1n})\}_{n=1}^{\infty}$, take a cluster point $k^*_2$ of this sequence and extract a subsequence $\{T_{2n}\}_{n=1}^{\infty}$ from the sequence $\{T_{1n}\}_{n=1}^{\infty}$ such that $T_{21} > 1$ and $\{k^*_2(T_{2n})\}_{n=1}^{\infty}$ converges to $k^*_2$. This procedure continues ad infinitum.

Finally, consider the sequence $\{c^*_t, s^*_t, k^*_{t+1}, r^*_{t+1}, w^*_{t+1}\}_{t=0}^{\infty}$, where $\{k^*_{t+1}\}_{t=0}^{\infty}$ is obtained by the diagonal procedure described above and for all $\tau \geq 0$,

$$c^*_\tau = f(k^*_\tau) - k^*_{\tau+1}, \quad s^*_\tau = k^*_{\tau+1}, \quad 1 + r^*_\tau = f'(k^*_\tau), \quad w^*_\tau = f(k^*_\tau) - f'(k^*_\tau)k^*_\tau. \tag{14}$$

The following lemma shows that this sequence is a SEP under perfect foresight.

**Lemma 2.3.** *The sequence $\{c^*_t, s^*_t, k^*_{t+1}, r^*_{t+1}, w^*_{t+1}\}_{t=0}^{\infty}$ is a sliding equilibrium path under perfect foresight starting from $s^*_{-1} = k^*_0$.*

*Proof.* See Appendix C. □

Thus we obtain a SEP under perfect foresight and prove the theorem. ■

Now we turn to the question of observational equivalence. The following theorem proves that a SEP under perfect foresight is observationally equivalent to some $\gamma$-optimal path only in the cases of log-utility and stationary sliding equilibria.

**Theorem 3.** *A sliding equilibrium path under perfect foresight starting from $s^*_{-1} \neq s^*$ is observationally equivalent to a $\gamma$-optimal path if and only if $\rho = 1$.*

*Proof.* Suppose that a SEP under perfect foresight, $\{c_t^*, s_t^*, k_{t+1}^*, r_t^*, w_t^*\}_{t=0}^{\infty}$, is observationally equivalent to some $\gamma$-optimal path. It follows from Theorem 1 that the equivalent discount factor must be $\gamma^*$, and $\{c_t^*, s_t^*, k_{t+1}^*, r_t^*, w_t^*\}_{t=0}^{\infty}$ converges to the SSE under perfect foresight $\{c^*, s^*, k^*, r^*, w^*\}$ characterized in Theorem 1.

Let $\Delta_{t+1}$ be given by

$$\Delta_{t+1} = \delta^{\frac{1}{\rho}}(1 + r_{t+2}^*)^{\frac{1-\rho}{\rho}} + \delta^{\frac{2}{\rho}}(1 + r_{t+2}^*)^{\frac{1-\rho}{\rho}}(1 + r_{t+3}^*)^{\frac{1-\rho}{\rho}} + \ldots . \tag{15}$$

It is easily seen that

$$\Delta_{t+1} = \delta^{\frac{1}{\rho}}(1 + r_{t+2}^*)^{\frac{1-\rho}{\rho}}(1 + \Delta_{t+2}) . \tag{16}$$

**Lemma 3.1.** *Let $\{c_t^*, s_t^*, k_{t+1}^*, r_t^*, w_t^*\}_{t=0}^{\infty}$ be a SEP under perfect foresight. Then*

$$c_{t+1}^* = c_t^* \left(\beta\delta(1 + r_{t+1}^*)\right)^{\frac{1}{\rho}} \frac{1 + \Delta_{t+1}}{1 + \beta^{\frac{1}{\rho}}\Delta_{t+1}} . \tag{17}$$

*Proof.* See Appendix D. □

Due to observational equivalence, $c_{t+1}^* = c_t^* \left(\gamma^*(1 + r_{t+1}^*)\right)^{\frac{1}{\rho}}$. Taking account of (17), we obtain that for all $t \geq 0$,

$$\left(\frac{\gamma^*}{\beta\delta}\right)^{\frac{1}{\rho}} = \frac{1 + \Delta_{t+1}}{1 + \beta^{\frac{1}{\rho}}\Delta_{t+1}}, \qquad \text{and hence} \qquad \Delta_{t+1} = \frac{(\gamma^*)^{\frac{1}{\rho}} - (\beta\delta)^{\frac{1}{\rho}}}{(\beta\delta)^{\frac{1}{\rho}} - (\beta\gamma^*)^{\frac{1}{\rho}}}.$$

Therefore, the value of $\Delta_{t+1}$ is constant over time. Using (16) and (7), we can restate this condition in terms of interest rates as follows:

$$(1 + r_{t+2}^*)^{\frac{1-\rho}{\rho}} = \frac{1}{(\gamma^*)^{\frac{1}{\rho}}} \cdot \frac{(\gamma^*)^{\frac{1}{\rho}} - (\beta\delta)^{\frac{1}{\rho}}}{\delta^{\frac{1}{\rho}} - (\beta\delta)^{\frac{1}{\rho}}} = (\gamma^*)^{1-\frac{1}{\rho}},$$

or

$$\left(\gamma^*(1 + r_{t+2}^*)\right)^{\frac{1-\rho}{\rho}} = 1, \qquad t \geq 0 . \tag{18}$$

Clearly, (18) holds only in the following two cases. First, $\rho = 1$, which is the log-utility case, and it was shown in Claim 2 that observational equivalence holds.

Second, $r_t^* = r^*$, which is the stationary case with $s_{-1}^* = s^*$. Thus, in the Ramsey model with quasi-hyperbolic discounting and isoelastic utility, a SEP under perfect foresight starting from $s_{-1}^* \neq s^*$ cannot be observationally equivalent to an optimal path in the Ramsey model with exponential discounting unless $\rho = 1$. ■

The intuition behind Theorem 3 is as follows. Under perfect foresight the expected prices are not revised and remain the same at each date. Therefore, on a SEP under perfect foresight the consumption levels at two adjacent dates are linked and satisfy the "first-order conditions" (17). These conditions are compatible with the first-order conditions in the standard Ramsey model if and only if either an interest rate is constant or utility is logarithmic. Thus, a SEP under perfect foresight is observationally equivalent to some $\gamma$-optimal path only in two cases. First, in the case of log-utility considered in Claim 2 where the equivalent discount factor is given by $\gamma^* = \frac{\beta\delta}{1-\delta+\beta\delta}$. Second, in the case of a SSE under perfect foresight considered in Theorem 1 where the equivalent discount factor $\gamma^*$ satisfies equation (7). In all other cases, a SEP under perfect foresight is not observationally equivalent to any $\gamma$-optimal path.

## 4.5 Comparison

Finally, we compare the properties of SEPs under pseudo-perfect and perfect foresight in terms of saving rates, long-run capital stocks and consumption.

As we have seen, Claims 1 and 2 suggest that already in the simplest case where $u(c) = \ln c$ and $f(k) = k^\alpha$, SEPs under pseudo-perfect and perfect foresight differ. By Claim 1, the SEP under pseudo-perfect foresight is observationally equivalent to the $\gamma^\circ$-optimal path, where $\gamma^\circ = \frac{\beta\delta}{1-\alpha\delta+\alpha\beta\delta}$ (which depends on the technology parameter $\alpha$). By Claim 2, the SEP under perfect foresight is observationally equivalent to the $\gamma^*$-optimal path, where $\gamma^* = \frac{\beta\delta}{1-\delta+\beta\delta}$ (which does not depend on the production technology). Comparing the equivalent discount factors, it is easily seen that $\beta\delta < \gamma^\circ < \gamma^* < \delta$. Thus, the equivalent exponential discount factor is higher under perfect foresight than under pseudo-perfect foresight.

It is well known that for $u(c) = \ln c$ and $f(k) = k^\alpha$, the saving rate on the

$\gamma$-optimal path is constant and given by $\alpha\gamma$. Therefore, in the considered case the saving rate is always higher under perfect foresight than under pseudo-perfect foresight. It follows that if the two economies start from the same initial condition, then the capital stock at each date on the SEP under perfect foresight is higher than the capital stock on the SEP under pseudo-perfect foresight. Consequently, while initially consumption is higher on the SEP under pseudo-perfect foresight, starting from some date, the SEP under perfect foresight provides higher consumption. Therefore, the stationary capital stock and consumption level are higher under perfect foresight than under pseudo-perfect foresight. Loosely speaking, more rationality means more consumption in the stationary state.

The result about stationary states can be generalized — it also holds for SSEs when intertemporal elasticity of substitution in consumption is sufficiently high. The comparison of consumer optima at given constant and non-constant sequences of interest and wage rates, allows us to compare SSEs under pseudo-perfect and perfect foresight. The following theorem maintains that when $0 < \rho \leq 1$, a stationary capital stock is higher under perfect foresight than under pseudo-perfect foresight, irrespective of a production technology.

**Theorem 4.** *Let* $\{c^*, s^*, k^*, r^*, w^*\}$ *be a stationary sliding equilibrium under perfect foresight, and* $\{c^\circ, s^\circ, k^\circ, r^\circ, w^\circ\}$ *be a stationary sliding equilibrium under pseudo-perfect foresight. If preferences are such that* $0 < \rho \leq 1$*, then*

$$c^* > c^\circ, \qquad s^* > s^\circ, \qquad k^* > k^\circ, \qquad r^* < r^\circ, \qquad w^* > w^\circ .$$

*Proof.* For the formal proof, see Appendix E. The idea is to assume the opposite and obtain a contradiction. Suppose that $k^* \leq k^\circ$. It then follows that $r^\circ \leq r^*$ and $w^\circ \geq w^*$. Let $c(r^\circ, w^\circ)$ be the date-$\tau$ consumption in the date-$\tau$ consumer optimum starting from $s^\circ$ at given constant interest rate $r^\circ$ and wage rate $w^\circ$.

Since $c^\circ$ is consumption in a stationary sliding equilibrium (under pseudo-perfect foresight), it can be checked that $k^\circ \geq k^*$ implies $c(r^\circ, w^\circ) \geq c^\circ$. However, when $0 < \rho \leq 1$, it is easily seen from (5) that the date-$\tau$ consumption in the date-$\tau$ consumer optimum starting from $s_{\tau-1}$ at given $\{r_t\}_{t=\tau}^{\infty}$ and $\{w_t\}_{t=\tau}^{\infty}$

is monotonically increasing in $w_t$ and decreasing in $r_t$ for all $t \geq \tau + 1$. Therefore, $c(r^\circ, w^\circ)$ is lower than the date-$\tau$ consumption on the date-$\tau$ equilibrium path starting from $s^\circ$, which is $c^\circ$. This contradiction shows that for $0 < \rho \leq 1$ we must have $k^* > k^\circ$. It then follows that $s^* > s^\circ$, $r^* < r^\circ$, $w^* > w^\circ$, and $c^* = f(k^*) - k^* > f(k^\circ) - k^\circ = c^\circ$. ■

We conjecture that the above result also holds for $\rho > 1$. A large number of simulations with different values of $\beta$ and $\delta$ support this conjecture.

# 5 Conclusion

In this paper, we consider the neoclassical growth model with quasi-hyperbolic discounting and emphasize the following main points. First, quasi-hyperbolic discounting involves time inconsistency, irrespective of whether we consider optimal or equilibrium paths. To describe the behavior of an agent who is naive about their time inconsistency and revises their consumer optimum at each date, we introduce sliding optimal and sliding equilibrium paths. Under exponential discounting, sliding paths correspond to the usual optimal and equilibrium paths.

Second, sliding equilibrium paths critically depend on expectations, and we distinguish between pseudo-perfect foresight and perfect foresight. Under pseudo-perfect foresight, the expected sequences of interest and wage rates at each date $\tau$ coincide with those realized on the "temporary" date-$\tau$ equilibrium path. Under perfect foresight, the expected sequences of interest and wage rates at each date coincide with those realized on the sliding path. Thus, an agent with pseudo-perfect foresight at each date is naive both about their future preferences (revising consumer optimum) and future prices (revising expectations about prices). In contrast, an agent with perfect foresight is only partially naive — such an agent revises their consumer optimum, but correctly foresees prices on the sliding path.

Third, in general, there is no observational equivalence of sliding equilibrium paths, i.e., a consumption path on a sliding equilibrium path coincides with that in a standard Ramsey model only in very special cases. We note that observational equivalence of sliding equilibrium paths under pseudo-perfect foresight holds only in two cases: the stationary case and the case with log-utility and Cobb–Douglas production technology. We also prove the existence of a sliding equilibrium path under perfect foresight for a general isoelastic utility function, and show that observational equivalence holds if and only if either the interest rate is constant or utility is logarithmic. In all other cases, a sliding equilibrium path under perfect foresight does not coincide with any optimal path in a standard Ramsey model.

Fourth, even in the simplest cases, the implications of pseudo-perfect and perfect foresight differ. We compare sliding equilibria under pseudo-perfect and perfect foresight in terms of saving rates for the stationary case and for the case with log-utility and Cobb–Douglas production technology. We show that less naivety leads to higher capital accumulation in the long run: perfect foresight implies a higher capital stock and a higher consumption level than pseudo-perfect foresight.

There are several open questions to be addressed by future studies. The dynamics of sliding equilibrium paths under perfect foresight is of interest, i.e., whether or not they converge to a stationary sliding equilibrium. Also, the conditions under which observational equivalence holds for sliding equilibrium paths under pseudo-perfect foresight are yet to be fully characterized.

Our research can be extended in a number of ways. It seems worthwhile to compare sliding equilibrium paths under pseudo-perfect and perfect foresight in terms of welfare, which is not an obvious task, as welfare criteria under time inconsistency are not clearly defined. Another possible direction of future research is to consider sliding equilibrium paths in a model where agents are heterogeneous in their time preferences or differ in their degree of naivety (i.e., some agents have pseudo-perfect foresight, and some have perfect foresight).

This paper clarifies the nature of time-inconsistent behavior under quasi-hyperbolic discounting in the neoclassical growth model. Sliding equilibrium paths under perfect foresight, though, can be used in a much wider range of applications. We believe that our approach will be useful for studying many other problems related to time-inconsistent decision making.

# Appendix

## A Characterizarion of sliding optimal paths

Let $V_\gamma(k)$ be the value function of problem (1):

$$V_\gamma(k) = \max_{0 \le k' \le f(k)} \; u(f(k) - k') + \gamma V_\gamma(k') \,. \tag{A.1}$$

The associated policy function $g_\gamma(k)$ is implicitly given by the functional equation:

$$u'(f(k) - g_\gamma(k)) = \gamma V'_\gamma(g_\gamma(k)) \,. \tag{A.2}$$

Thus the $\gamma$-optimal path in the standard Ramsey model is fully determined by value function $V_\gamma(k)$ or policy function $g_\gamma(k)$.

Let us characterize the first step in problem (6). Since the continuation of the date-$\tau$ optimal path is the $\delta$-optimal path, it follows that at date $\tau$ the quasi-hyperbolic planner solves the following problem:

$$\max_{0 \le k' \le f(k)} \; u(f(k) - k') + \beta\delta V_\delta(k') \,.$$

Therefore, the policy function $h(k)$ which solves the functional equation

$$u'(f(k) - h(k)) = \beta\delta V'_\delta(h(k)) \,, \tag{A.3}$$

describes the first step on the date-$\tau$ optimal path, and hence determines a SOP which is a sequence $\{c^\circ_t, k^\circ_{t+1}\}_{t=0}^\infty$ such that $c^\circ_t = f(k^\circ_t) - h(k^\circ_t)$ and $k^\circ_{t+1} = h(k^\circ_t)$.

Clearly, a SOP under $\beta$–$\delta$ discounting is observationally equivalent to some $\gamma$-optimal path if and only if $h(k) = g_\gamma(k)$ for all $k$, which, using (A.2) and (A.3), can be written as

$$\gamma V'_\gamma(k) = \beta\delta V'_\delta(k), \qquad \forall k \,. \tag{A.4}$$

Therefore, observational equivalence of SOPs depends on the properties of a value function of the standard Ramsey model.

It is very hard to expect that the derivative of a value function, $\gamma V'_\gamma(k)$, simply scales when a discount factor changes, i.e., that equation (A.4) holds. To the best of our knowledge, a SOP under quasi-hyperbolic discounting is observationally equivalent to a $\gamma$-optimal path either for log-utility and Cobb–Douglas production technology or for a stationary sliding optimum.

*Proof of Claim 1.* When $u(c) = \ln c$ and $f(k) = k^\alpha$, there is a closed-form solution for (A.1): $V_\gamma(k) = \frac{1}{1-\gamma}\Big(\ln(1-\alpha\gamma) + \frac{\alpha\gamma}{1-\alpha\gamma}\ln(\alpha\gamma)\Big) + \frac{\alpha}{1-\alpha\gamma}\ln k$, and the associated policy function is given by $g_\gamma(k) = \alpha\gamma k^\alpha$. Then equation (A.3) which determines the policy function $h(k)$ associated with a SOP takes the form $\frac{1}{k^\alpha - h(k)} = \beta\delta V'_\delta(h(k))$, which can be rewritten as $\frac{1}{k^\alpha - h(k)} = \frac{\alpha\beta\delta}{1-\alpha\delta}\,\frac{1}{h(k)}$, so that $h(k) = \frac{\alpha\beta\delta}{1-\alpha\delta+\alpha\beta\delta}k^\alpha$. Hence $h(k)$ coincides with the policy function $g_{\gamma^\circ}(k)$ associated with the $\gamma^\circ$-optimal path for $\gamma^\circ = \frac{\beta\delta}{1-\alpha\delta+\alpha\beta\delta}$. Therefore, a SOP is observationally equivalent to $\gamma^\circ$-optimal path. ■

A stationary sliding optimum is observationally equivalent to a stationary $\gamma^\circ$-optimum for $\gamma^\circ = 1/f'(k^\circ)$. Let us show that $\gamma^\circ < \delta$, which is equivalent to $k^\circ < k^\delta$, where $k^\delta$ is the modified golden rule capital stock for the discount factor $\delta$. Indeed, $k^\circ$ is the solution to equation (A.3) for $h(k) = k$:

$$f(k) - k \;=\; \left(\beta\delta V'_\delta(k)\right)^{-\frac{1}{\rho}}. \tag{A.5}$$

Consider the functions $L(k) = f(k) - k$ and $R(k) = \frac{1}{\left(\beta\delta V'_\delta(k)\right)^{\frac{1}{\rho}}}$. Clearly, $L(k)$ monotonically increases for $k < k^\delta$, is concave, and $L(0) = 0$. At the same time, since $V_\delta(k)$ is concave, we have

$$R'(k) \;=\; \frac{1}{\rho(\beta\delta)^{\frac{1}{\rho}}}\frac{|V''_\delta(k)|}{\left(V'_\delta(k)\right)^{1+\frac{1}{\rho}}} \;>\; 0,$$

Thus $R(k)$ also monotonically increases for $k < k^\delta$, and $R(0) = 0$. When $\beta = 1$, the solution to (A.5) is $k^\delta$. As $\beta$ decreases, $R(k)$ shifts upward, and hence the capital stock which solves equation (A.5) decreases. This means that for $\beta < 1$, we have $k^\circ < k^\delta$, i.e., $\gamma^\circ < \delta$.

# B Proof of Theorem 1

## B.1 Proof of Lemma 1.1

Since $\frac{1}{1+r^*} < 1$, the right-hand side of the budget constraint in problem (4) under constant interest rate $r^*$ and wage rate $w^*$ is finite:

$$(1+r^*)s^* + \sum_{t=0}^{\infty} \frac{w^*}{(1+r^*)^t} = (1+r^*)s^* + \frac{w^*}{1-\frac{1}{1+r^*}} = \frac{1+r^*}{r^*}(r^*s^* + w^*) < +\infty.$$

It now follows from (5) that

$$c_\tau^{**\tau} = \frac{\frac{1+r^*}{r^*}(r^*s^*+w^*)}{1+(\beta\delta)^{\frac{1}{\rho}}(1+r^*)^{\frac{1-\rho}{\rho}} + \ldots + (\beta\delta^t)^{\frac{1}{\rho}}(1+r^*)^{\frac{t(1-\rho)}{\rho}} + \ldots}.$$

Since $\delta(1+r^*)^{1-\rho} < 1$, the sum in the denominator is finite:

$$\begin{aligned} 1 + (\beta\delta)^{\frac{1}{\rho}}(1+r^*)^{\frac{1-\rho}{\rho}} \left(1 + (\delta(1+r^*)^{1-\rho})^{\frac{1}{\rho}} + (\delta(1+r^*)^{1-\rho})^{\frac{2}{\rho}} + \ldots\right) \\ = 1 + \frac{(\beta\delta)^{\frac{1}{\rho}}(1+r^*)^{\frac{1-\rho}{\rho}}}{1-\delta^{\frac{1}{\rho}}(1+r^*)^{\frac{1-\rho}{\rho}}} = \frac{1-\delta^{\frac{1}{\rho}}(1+r^*)^{\frac{1-\rho}{\rho}} + (\beta\delta)^{\frac{1}{\rho}}(1+r^*)^{\frac{1-\rho}{\rho}}}{1-\delta^{\frac{1}{\rho}}(1+r^*)^{\frac{1-\rho}{\rho}}}. \end{aligned}$$

Thus a date-$\tau$ consumer optimum at given $r^*$ and $w^*$ exists and is unique. In this optimum, $c_\tau^{**\tau}$ is given by (8).

## B.2 Proof of Lemma 1.2

The interest rate on a SSE under perfect foresight, $r^*$, satisfies

$$1 + r^* = \frac{1-\delta^{\frac{1}{\rho}}(1+r^*)^{\frac{1-\rho}{\rho}} + (\beta\delta)^{\frac{1}{\rho}}(1+r^*)^{\frac{1-\rho}{\rho}}}{(\beta\delta)^{\frac{1}{\rho}}(1+r^*)^{\frac{1-\rho}{\rho}}}.$$

Rearranging the above equation, we get

$$(\beta\delta)^{\frac{1}{\rho}}(1+r^*)^{\frac{1}{\rho}} = 1-\delta^{\frac{1}{\rho}}(1+r^*)^{\frac{1-\rho}{\rho}} + (\beta\delta)^{\frac{1}{\rho}}(1+r^*)^{\frac{1-\rho}{\rho}},$$

and hence $\frac{1}{1+r^*} = \frac{\left(\frac{1}{1+r^*}\right)^{\frac{1}{\rho}} - (\beta\delta)^{\frac{1}{\rho}}}{\delta^{\frac{1}{\rho}} - (\beta\delta)^{\frac{1}{\rho}}}$. Thus $r^*$ is the solution to equation (9).

Denote $\gamma^* = \frac{1}{1+r^*}$. It is clear that equations (9) and (7) are essentially the same. Let us show that there exists a unique solution to equation (7), and this solution satisfies $\delta < (\gamma^*)^{1-\rho}$, i.e., $\delta(1+r^*)^{1-\rho} < 1$.

Consider the functions $L(\gamma) = \gamma$, and $R(\gamma) = \frac{\gamma^{\frac{1}{\rho}} - (\beta\delta)^{\frac{1}{\rho}}}{\delta^{\frac{1}{\rho}} - (\beta\delta)^{\frac{1}{\rho}}}$. Both $L(\gamma)$ and $R(\gamma)$ are monotonically increasing in $\gamma$. Moreover, $R(\beta\delta) = 0 < \beta\delta = L(\beta\delta)$, and $R(\delta) = 1 > \delta = L(\delta)$. Since $R(\gamma)$ is strictly convex for $\rho < 1$, linear for $\rho = 1$, and strictly concave for $\rho > 1$, for any $\rho$ there is a unique $\gamma^*$ such that $L(\gamma^*) = R(\gamma^*)$. This $\gamma^*$ is the solution to equation (7), and it is clear that $\beta\delta < \gamma^* < \delta$.

Since $\rho > 0$, we also have $(\beta\delta)^{\frac{1}{\rho}} < (\gamma^*)^{\frac{1}{\rho}} < \delta^{\frac{1}{\rho}}$, and hence

$$\gamma^* = \frac{(\gamma^*)^{\frac{1}{\rho}} - (\beta\delta)^{\frac{1}{\rho}}}{\delta^{\frac{1}{\rho}} - (\beta\delta)^{\frac{1}{\rho}}} < \frac{(\gamma^*)^{\frac{1}{\rho}}}{\delta^{\frac{1}{\rho}}}.$$

Therefore, $\delta < (\gamma^*)^{1-\rho}$, which completes the proof of the lemma.

# C Proof of Theorem 2

## C.1 A $T$-horizon SEP under perfect foresight

Fix a finite horizon $T > 0$ and consider a $T$-horizon date-$\tau$ consumer optimum. Similarly to the infinite horizon case (cf. equation (5)), it is easily checked that for all $0 \leq \tau \leq T$, date-$\tau$ consumption level on a $T$-horizon date-$\tau$ consumer optimum starting from $k_\tau$ at given $\{r_{t+1}\}_{t=\tau}^T$ and $\{w_{t+1}\}_{t=\tau}^T$ satisfies

$$c_\tau^\tau = \frac{f(k_\tau) + \sum_{t=\tau+1}^{T+1} \frac{w_t}{(1+r_{\tau+1})\cdots(1+r_t)}}{1 + (\beta\delta)^{\frac{1}{\rho}}(1+r_{\tau+1})^{\frac{1-\rho}{\rho}} + \ldots + (\beta\delta^{T+1-\tau})^{\frac{1}{\rho}}\left((1+r_{\tau+1})\cdots(1+r_{T+1})\right)^{\frac{1-\rho}{\rho}}}.$$

Also, similarly to the infinite horizon case, a $T$-horizon SEP under perfect foresight is associated with $T+1$ optimization problems of the form (10), i.e., with the sequence of the corresponding $T$-horizon consumer optima. More precisely, a $T$-horizon SEP under perfect foresight starting from $s_{-1}^* = k_0^*$ is a sequence $\{c_t^*, s_t^*, k_{t+1}^*, r_{t+1}^*, w_{t+1}^*\}_{t=0}^T$ characterized as follows:

- there exists a $T$-horizon date-0 consumer optimum starting from $k_0^*$ at given

$\{r^*_{t+1}\}_{t=0}^T$ and $\{w^*_{t+1}\}_{t=0}^T$, which we denote by $\{c_t^{**0}, s_t^{**0}\}_{t=0}^{T+1}$, and its first elements are precisely the date-0 consumption and savings on a $T$-horizon SEP under perfect foresight: $c_0^{**0} = c_0^*$ and $s_0^{**0} = s_0^*$;

- there exists a $T$-horizon date-1 consumer optimum starting from $k_1^*$ at given $\{r^*_{t+1}\}_{t=1}^T$ and $\{w^*_{t+1}\}_{t=1}^T$, denoted by $\{c_t^{**1}, s_t^{**1}\}_{t=1}^{T+1}$, and its first elements are the date-1 consumption and savings on a $T$-horizon SEP under perfect foresight: $c_1^{**1} = c_1^*$ and $s_1^{**1} = s_1^*$;
- and so on, till the $T$-horizon date-$T$ consumer optimum starting from $k_T^*$ given $r^*_{T+1}$ and $w^*_{T+1}$, denoted by $\{c_t^{**T}, s_t^{**T}\}_{t=T}^{T+1}$, whose first elements are the date-$T$ consumption and savings on a $T$-horizon SEP under perfect foresight: $c_T^{**T} = c_T^*$ and $s_T^{**T} = s_T^*$;
- so that the resulting sliding equilibrium capital stock sequence $\{k^*_{t+1}\}_{t=0}^T = \{s_t^{**t}\}_{t=0}^T$ determines precisely those interest and wage rates which were correctly expected by the agent when solving for consumer optima at each date.

## C.2 Proof of Lemma 2.1

**Step 1. Lower bounds for capital.**

Let $\{\underline{c}_t(T), \underline{k}_{t+1}(T)\}_{t=0}^{T+1}$ be a solution to problem (11). Let for $0 \le t \le T+1$, $\underline{s}_t(T) = \underline{k}_{t+1}(T)$, $1 + \underline{r}_t(T) = f'(\underline{k}_t(T))$, and $\underline{w}_t(T) = f(\underline{k}_t(T)) - f'(\underline{k}_t(T))\underline{k}_t(T)$. It is well known that the sequence $\{\underline{c}_t(T), \underline{s}_t(T), \underline{k}_{t+1}(T), \underline{r}_t(T), \underline{w}_t(T)\}_{t=0}^{T+1}$ is a $T$-horizon equilibrium path starting from $\underline{s}_{-1} = \underline{k}_0$ in the standard Ramsey model with the discount factor $\beta\delta$. In what follows, a finite horizon $T$ is fixed, so we shall omit the notation "$(T)$" as long as it does not lead to confusion.

In particular, $\{\underline{c}_t\}_{t=0}^{T+1}$ is a solution to the problem

$$\max_{c_t \ge 0} \sum_{t=0}^{T+1} (\beta\delta)^t u(c_t), \qquad \text{s. t. } c_0 + \sum_{t=1}^{T+1} \frac{c_t}{(1+\underline{r}_1)\cdots(1+\underline{r}_t)} \le f(\underline{k}_0) + \sum_{t=1}^{T+1} \frac{\underline{w}_t}{(1+\underline{r}_1)\cdots(1+\underline{r}_t)},$$

and the following observation holds.

**Claim C.1.** *For all* $0 \leq \tau \leq T$,

$$\underline{c}_\tau = \frac{f(\underline{k}_\tau) + \sum_{t=1}^{T+1} \frac{\underline{w}_t}{(1+\underline{r}_1)\cdots(1+\underline{r}_t)}}{1 + (\beta\delta)^{\frac{1}{\rho}}(1+\underline{r}_{\tau+1})^{\frac{1-\rho}{\rho}} + \ldots + (\beta\delta)^{\frac{T+1-\tau}{\rho}}\left((1+\underline{r}_{\tau+1})\cdots(1+\underline{r}_{T+1})\right)^{\frac{1-\rho}{\rho}}}.$$

*Proof.* The solution $\{\underline{c}_t\}_{t=0}^{T+1}$ satisfies the following first-order conditions:

$$\underline{c}_{t+1} = \left(\beta\delta(1+\underline{r}_{t+1})\right)^{\frac{1}{\rho}} \underline{c}_t, \qquad 0 \leq t \leq T. \tag{C.1}$$

Substituting (C.1) into the budget constraint which holds as equality, we get

$$\underline{c}_0 \left(1 + \sum_{t=1}^{T+1} (\beta\delta)^{\frac{t}{\rho}} \left((1+\underline{r}_1)\cdots(1+\underline{r}_t)\right)^{\frac{1-\rho}{\rho}}\right) = f(\underline{k}_0) + \sum_{t=1}^{T+1} \frac{\underline{w}_t}{(1+\underline{r}_1)\cdots(1+\underline{r}_t)},$$

which proves the claim for $\tau = 0$. To prove it for $1 \leq \tau \leq T$, it is sufficient to note that the corresponding budget constraint holds for any $\tau$:

$$\underline{c}_\tau + \sum_{t=\tau+1}^{T+1} \frac{\underline{c}_t}{(1+\underline{r}_{\tau+1})\cdots(1+\underline{r}_t)} = f(\underline{k}_\tau) + \sum_{t=\tau+1}^{T+1} \frac{\underline{w}_t}{(1+\underline{r}_{\tau+1})\cdots(1+\underline{r}_t)}, \tag{C.2}$$

and repeat the argument. ■

**Step 2. Constructing a fixed point.**

Consider the following set consisting of bounded sequences of length $T+1$:

$$\mathcal{S}_T = \left\{ \{k_{t+1}\}_{t=0}^T \;\middle|\; \underline{k}_1 \leq k_1 \leq f(k_0^*), \;\text{ and }\; \underline{k}_{t+1} \leq k_{t+1} \leq f(k_t), \;\; 1 \leq t \leq T \right\}.$$

Clearly, $\mathcal{S}_T$ is a non-empty compact convex set. Now, given $\{k_{t+1}\}_{t=0}^T \in \mathcal{S}_T$, consider the sequence $\{\tilde{k}_{t+1}\}_{t=0}^T$ constructed recursively as follows:

$$\tilde{k}_1 = \max\{\underline{k}_1,\; f(k_0^*) - c_0^0\}, \qquad \tilde{k}_{\tau+1} = \max\{\underline{k}_{\tau+1},\; f(\tilde{k}_\tau) - c_\tau^\tau\}, \quad 1 \leq \tau \leq T,$$

where $c_\tau^\tau$ is the date-$\tau$ consumption on a $T$-horizon date-$\tau$ consumer optimum starting from $k_\tau$ at given interest and wage rates determined by the sequence

$\{k_{t+1}\}_{t=\tau}^{T}$ as $1 + r_t = f'(k_t)$, and $w_t = f(k_t) - f'(k_t)k_t$.

By construction, we have $\tilde{k}_{t+1} \geq \underline{k}_{t+1}$ for all $0 \leq t \leq T$. Moreover, since $c_\tau^\tau \geq 0$ and $\underline{k}_{t+1} \leq f(\underline{k}_t)$, it also follows that $\tilde{k}_1 \leq f(k_0^*)$ and $\tilde{k}_{t+1} \leq f(\tilde{k}_t)$ for $1 \leq t \leq T$. Therefore, $\{\tilde{k}_{t+1}\}_{t=0}^{T} \in \mathcal{S}_T$.

Thus we have a continuous mapping from a compact convex set $\mathcal{S}_T$ to itself such that $\{k_{t+1}\}_{t=0}^{T}$ maps into $\{\tilde{k}_{t+1}\}_{t=0}^{T}$. By the Brouwer's fixed point theorem, there is a sequence $\{k_{t+1}^*\}_{t=0}^{T} \in \mathcal{S}_T$, which is a fixed point of this mapping.

**Step 3. Useful claims**.

For the obtained fixed point $\{k_{t+1}^*\}_{t=0}^{T}$, let $\{r_{t+1}^*\}_{t=0}^{T}$ and $\{w_{t+1}^*\}_{t=0}^{T}$ be given by $1 + r_t^* = f'(k_t^*)$ and $w_t^* = f(k_t^*) - f'(k_t^*)k_t^*$ for all $0 \leq t \leq T$. Let $\{c_t^{**\tau}, s_t^{**\tau}\}_{t=\tau}^{T+1}$ be a $T$-horizon date-$\tau$ consumer optimum starting from $k_\tau^*$ at given $\{r_{t+1}^*\}_{t=\tau}^{T}$ and $\{w_{t+1}^*\}_{t=\tau}^{T}$. Denote the date-$\tau$ consumption on this $T$-horizon date-$\tau$ consumer optimum by $c_\tau^* = c_\tau^{**\tau}$. For all $0 \leq \tau \leq T$ (cf. Section C.1),

$$c_\tau^* = \frac{f(k_\tau^*) + \sum_{t=\tau+1}^{T+1} \frac{w_t^*}{(1+r_{\tau+1}^*)\cdots(1+r_t^*)}}{1 + (\beta\delta)^{\frac{1}{\rho}}(1 + r_{\tau+1}^*)^{\frac{1-\rho}{\rho}} + \ldots + (\beta\delta^{T+1-\tau})^{\frac{1}{\rho}}\left((1 + r_{\tau+1}^*)\cdots(1 + r_{T+1}^*)\right)^{\frac{1-\rho}{\rho}}}. \tag{C.3}$$

The following claim establishes a useful property of the sequence $\{c_t^*\}_{t=0}^{T}$.

**Claim C.2.** *For* $0 \leq t \leq T - 1$,

$$c_{t+1}^* \;\geq\; c_t^* \left(\beta\delta(1 + r_{t+1}^*)\right)^{\frac{1}{\rho}} \frac{1 + \Delta_t^T}{1 + \beta^{\frac{1}{\rho}}\Delta_t^T} \qquad (= \;\textit{if}\;\; k_{t+1}^* > \underline{k}_{t+1}), \tag{C.4}$$

*where* $\Delta_t^T = \delta^{\frac{1}{\rho}}(1 + r_{t+2}^*)^{\frac{1-\rho}{\rho}} + \ldots + \delta^{\frac{T+1-t}{\rho}}(1 + r_{t+2}^*)^{\frac{1-\rho}{\rho}} \cdots (1 + r_{T+1}^*)^{\frac{1-\rho}{\rho}}$.

*Proof.* Note that for $0 \leq t \leq T - 2$, we have $\Delta_t^T = \delta^{\frac{1}{\rho}}(1 + r_{t+2}^*)^{\frac{1-\rho}{\rho}}\left(1 + \Delta_{t+1}^T\right)$ (cf. equation (16)). Then (C.3) can be rewritten as

$$c_t^*\left(1 + (\beta\delta)^{\frac{1}{\rho}}(1 + r_{t+1}^*)^{\frac{1-\rho}{\rho}}(1 + \Delta_t^T)\right) = f(k_t^*) + \sum_{s=t+1}^{T+1} \frac{w_s^*}{(1 + r_{t+1}^*)\cdots(1 + r_s^*)}. \tag{C.5}$$

By construction of $k_{\tau+1}^*$, we have $c_\tau^* + k_{\tau+1}^* \geq f(k_\tau^*)$ (= if $k_{\tau+1}^* > \underline{k}_{\tau+1}$), and hence it follows from (C.5) for $t = \tau$ that

$$c^*_\tau(\beta\delta)^{\frac{1}{\rho}}(1+r^*_{\tau+1})^{\frac{1-\rho}{\rho}}(1+\Delta^T_\tau) \;\leq\; k^*_{\tau+1} + \sum_{t=\tau+1}^{T+1} \frac{w^*_t}{(1+r^*_{\tau+1})\cdots(1+r^*_t)}, \qquad \text{(C.6)}$$

(with equality when $k^*_{\tau+1} > \underline{k}_{\tau+1}$).

Consider the value $c^*_{\tau+1}\left(1+\beta^{\frac{1}{\rho}}\Delta^T_\tau\right)$. By (C.5) for $t = \tau+1$ and (C.6), we get

$$c^*_{\tau+1}\left(1+\beta^{\frac{1}{\rho}}\Delta^T_\tau\right) = c^*_{\tau+1}\left(1+(\beta\delta)^{\frac{1}{\rho}}(1+r^*_{\tau+2})^{\frac{1-\rho}{\rho}}(1+\Delta^T_{\tau+1})\right)$$

$$= (1+r^*_{\tau+1})k^*_{\tau+1} + w^*_{\tau+1} + \sum_{t=\tau+2}^{T+1} \frac{w^*_t}{(1+r^*_{\tau+2})\cdots(1+r^*_t)}$$

$$= (1+r^*_{\tau+1})\left(k^*_{\tau+1} + \sum_{t=\tau+1}^{T+1} \frac{w^*_t}{(1+r^*_{\tau+1})\cdots(1+r^*_t)}\right) \;\geq\; c^*_\tau(\beta\delta)^{\frac{1}{\rho}}(1+r^*_{\tau+1})^{\frac{1}{\rho}}(1+\Delta^T_\tau)$$

(again, with equality when $k^*_{\tau+1} > \underline{k}_{\tau+1}$).

Therefore, the consumption levels $c^*_{\tau+1}$ and $c^*_\tau$ are linked as follows:

$$c^*_{\tau+1} \;\geq\; c^*_\tau\left(\beta\delta(1+r^*_{\tau+1})\right)^{\frac{1}{\rho}}\frac{1+\Delta^T_\tau}{1+\beta^{\frac{1}{\rho}}\Delta^T_\tau}, \qquad (= \text{ if } \; k^*_{\tau+1} > \underline{k}_{\tau+1}),$$

and this holds true for all $0 \leq \tau \leq T-1$. ■

Claim C.2 allows us to prove that the lower bound in $\mathcal{S}_T$ is never binding for our fixed point.

**Claim C.3.** *For all* $0 \leq t \leq T$*, we have* $k^*_{t+1} > \underline{k}_{t+1}$.

*Proof.* First, let us prove that $k^*_{t+1} > \underline{k}_{t+1}$ for all $t < T$. Suppose the opposite, i.e., that $k^*_{\tau+1} = \underline{k}_{\tau+1}$ for some $\tau \leq T-1$. Then, by construction of $k^*_{\tau+1}$,

$$c^*_\tau \;\geq\; f(k^*_\tau) - k^*_{\tau+1} \;\geq\; f(\underline{k}_\tau) - k^*_{\tau+1} \;=\; f(\underline{k}_\tau) - \underline{k}_{\tau+1} \;=\; \underline{c}_\tau. \qquad \text{(C.7)}$$

Let us show that in this case $k^*_{\tau+2} = \underline{k}_{\tau+2}$. Indeed, suppose the opposite, i.e., $k^*_{\tau+2} > \underline{k}_{\tau+2}$. Then, by construction of $k^*_{\tau+2}$,

$$c^*_{\tau+1} = f(k^*_{\tau+1}) - k^*_{\tau+2} = f(\underline{k}_{\tau+1}) - k^*_{\tau+2} < f(\underline{k}_{\tau+1}) - \underline{k}_{\tau+2} = \underline{c}_{\tau+1}.$$

However, since $1+\Delta_\tau^T > 1+\beta^{\frac{1}{\rho}}\Delta_\tau^T$, it follows from (C.4), (C.7) and (C.1) that

$$c^*_{\tau+1} \geq c^*_\tau \left(\beta\delta(1+r^*_{\tau+1})\right)^{\frac{1}{\rho}} \frac{1+\Delta_\tau^T}{1+\beta^{\frac{1}{\rho}}\Delta_\tau^T} > \left(\beta\delta(1+r^*_{\tau+1})\right)^{\frac{1}{\rho}} c^*_\tau \geq \left(\beta\delta(\underline{r}_{\tau+1})\right)^{\frac{1}{\rho}} \underline{c}_\tau = \underline{c}_{\tau+1}.$$

This contradiction shows that $k^*_{\tau+2} = \underline{k}_{\tau+2}$. Similarly to (C.7), it follows that

$$c^*_{\tau+1} \;\geq\; f(k^*_{\tau+1}) - k^*_{\tau+2} \;=\; f(\underline{k}_{\tau+1}) - \underline{k}_{\tau+2} \;=\; \underline{c}_{\tau+1}. \tag{C.8}$$

Repeating the argument, we obtain that if $k^*_{\tau+1} = \underline{k}_{\tau+1}$, then $k^*_{t+1} = \underline{k}_{t+1}$ for all $\tau+1 \leq t \leq T$, and hence the sequence $\{k^*_{t+1}\}_{t=\tau}^T$ coincides with the sequence $\{\underline{k}_{t+1}\}_{t=\tau}^T$. However, in this case for all $\tau \leq t \leq T$, we have $1+r^*_{t+1} = 1+\underline{r}_{t+1}$ and $w^*_{t+1} = \underline{w}_{t+1}$, and it follows from Claim C.1 and (C.3) that

$$\underline{c}_{\tau+1} = \frac{f(k^*_{\tau+1}) + \sum_{t=\tau+2}^{T+1} \frac{w^*_t}{(1+r^*_{\tau+2})\cdots(1+r^*_t)}}{1+(\beta\delta)^{\frac{1}{\rho}}(1+r^*_{\tau+2})^{\frac{1-\rho}{\rho}} + \ldots + (\beta\delta)^{\frac{T+1-\tau}{\rho}}\left((1+r^*_{\tau+2})\cdots(1+r^*_{T+1})\right)^{\frac{1-\rho}{\rho}}} >$$
$$\frac{f(k^*_{\tau+1}) + \sum_{t=\tau+2}^{T+1} \frac{w^*_t}{(1+r^*_{\tau+2})\cdots(1+r^*_t)}}{1+(\beta\delta)^{\frac{1}{\rho}}(1+r^*_{\tau+2})^{\frac{1-\rho}{\rho}} + \ldots + (\beta\delta^{T+1-\tau})^{\frac{1}{\rho}}\left((1+r^*_{\tau+2})\cdots(1+r^*_{T+1})\right)^{\frac{1-\rho}{\rho}}} = c^*_{\tau+1},$$

because in the denominator $\beta < 1$. This contradiction to (C.8) shows that for all $0 \leq t \leq T-1$, we have $k^*_{t+1} > \underline{k}_{t+1}$.

Second, let us prove that $k^*_{T+1} > \underline{k}_{T+1}$. Suppose the opposite, i.e., $k^*_{T+1} = \underline{k}_{T+1}$, and hence $1+r^*_{T+1} = 1+\underline{r}_{T+1}$ and $w^*_{T+1} = \underline{w}_{T+1}$. By construction of $k^*_{T+1}$,

$$c^*_T \;\geq\; f(k^*_T) - k^*_{T+1} \;\geq\; f(\underline{k}_T) - k^*_{T+1} \;=\; f(\underline{k}_T) - \underline{k}_{T+1} \;=\; \underline{c}_T. \tag{C.9}$$

Consider now the $T$-horizon date-$T$ consumer optimum starting from $k^*_T$ at given $1+r^*_{T+1}$ and $w^*_{T+1}$. By the first-order condition, (C.9) and (C.1), we have $c^{**T}_{T+1} \;=\; \left(\beta\delta(1+r^*_{T+1})\right)^{\frac{1}{\rho}} c^*_T \;=\; \left(\beta\delta(1+\underline{r}_{T+1})\right)^{\frac{1}{\rho}} c^*_T \;>\; \left(\beta\delta(1+\underline{r}_{T+1})\right)^{\frac{1}{\rho}} \underline{c}_T \;=\; \underline{c}_{T+1}$. At the same time, it follows from the budget constraint, (C.9) and (C.2) that

$$\begin{aligned} c^{**T}_{T+1} \;&=\; (1+r^*_{T+1})\left(f(k^*_T) - c^*_T\right) + w^*_{T+1} \;\leq\; (1+r^*_{T+1})k^*_{T+1} + w^*_{T+1} \\ &= f(k^*_{T+1}) \;=\; f(\underline{k}_{T+1}) = (1+\underline{r}_{T+1})\left(f(\underline{k}_T) - \underline{c}_T\right) + \underline{w}_{T+1} \;=\; \underline{c}_{T+1}. \end{aligned}$$

This contradiction shows that $k^*_{T+1} > \underline{k}_{T+1}$, which proves the claim. ■

**Step 4. Ensuring existence.**

Claim C.3 shows that for all $0 \le \tau \le T$, our obtained fixed point satisfies $k^*_{\tau+1} > \underline{k}_{\tau+1}$. Let for $0 \le \tau \le T$, $s^*_\tau = k^*_{\tau+1}$. Then $s^*_\tau = f(k^*_\tau) - c^*_\tau$, and hence by construction $c^*_\tau$ and $s^*_\tau$ are the first elements of the $T$-horizon date-$\tau$ consumer optimum starting from $k^*_\tau$ at given $\{r^*_{t+1}\}^T_{t=\tau}$ and $\{w^*_{t+1}\}^T_{t=\tau}$.

It follows that the obtained fixed point determines the $T$-horizon SEP under perfect foresight starting from $s^*_{-1} = k^*_0$. Formally, the sequence $\{c^*_t, s^*_t, k^*_{t+1}, r^*_{t+1}, w^*_{t+1}\}^T_{t=0}$, where $\{k^*_{t+1}\}^T_{t=0}$ is the fixed point of the described mapping $\mathcal{S}_T \to \mathcal{S}_T$, and for all $0 \le \tau \le T$, $c^*_\tau = f(k^*_\tau) - k^*_{\tau+1}$, $s^*_\tau = k^*_{\tau+1}$, $1 + r^*_\tau = f'(k^*_\tau)$, and $w^*_\tau = f(k^*_\tau) - f'(k^*_\tau)k^*_\tau$, satisfies Definition 9 and hence is a $T$-horizon SEP under perfect foresight.

## C.3 Proof of Lemma 2.2

It follows from Claim C.3 that the sequence of capital stocks on a $T$-horizon SEP under perfect foresight is bounded from below by the sequence of capital stocks $\{\underline{k}_{t+1}(T)\}^T_{t=0}$. It remains to show that it is also bounded from above.

**Step 1. Upper bounds for capital.**

Let $\{\bar{c}_t(T), \bar{k}_{t+1}(T)\}^{T+1}_{t=0}$ be a solution to problem (12). Let for $0 \le t \le T+1$, $\bar{s}_t(T) = \bar{k}_{t+1}(T)$, $1 + \bar{r}_t(T) = f'(\bar{k}_t(T))$, and $\bar{w}_t(T) = f(\bar{k}_t(T)) - f'(\bar{k}_t(T))\bar{k}_t(T)$. Then the sequence $\{\bar{c}_t(T), \bar{s}_t(T), \bar{k}_{t+1}(T), \bar{r}_t(T), \bar{w}_t(T)\}^{T+1}_{t=0}$ is a $T$-horizon equilibrium path starting from $\bar{s}_{-1} = \bar{k}_0$ in the standard Ramsey model with the discount factor $\delta$. In what follows we omit the notation "$(T)$", as there will be no confusion.

Similarly to (C.1) and (C.2), it is easily checked that $\{\bar{c}_\tau\}^{T+1}_{\tau=0}$ satisfies

$$\bar{c}_{\tau+1} = (\delta(1 + \bar{r}_{\tau+1}))^{\frac{1}{\rho}}\, \bar{c}_\tau, \tag{C.10}$$

$$\bar{c}_\tau + \sum_{t=\tau+1}^{T+1} \frac{\bar{c}_t}{(1 + \bar{r}_{\tau+1}) \cdots (1 + \bar{r}_t)} = f(\bar{k}_\tau) + \sum_{t=\tau+1}^{T+1} \frac{\bar{w}_t}{(1 + \bar{r}_{\tau+1}) \cdots (1 + \bar{r}_t)}. \tag{C.11}$$

**Step 2. Upper and lower bounds for the consumption growth rate.**

Let $\{c_t^*, s_t^*, k_{t+1}^*, r_{t+1}^*, w_{t+1}^*\}_{t=0}^T$ be a $T$-horizon SEP under perfect foresight starting from $s_{-1}^* = k_0^*$. We already know that date-$\tau$ consumption on this path is given by (C.3). The following claim provides upper and lower bounds for the consumption growth rate on a $T$-horizon SEP under perfect foresight.

**Claim C.4.** *On a $T$-horizon SEP under perfect foresight we have*

$$(\beta\delta(1+r_{t+1}^*))^{\frac{1}{\rho}} \;<\; \frac{c_{t+1}^*}{c_t^*} \;<\; (\delta(1+r_{t+1}^*))^{\frac{1}{\rho}}\,. \tag{C.12}$$

*Proof.* By Claims C.2 and C.3, on a $T$-horizon SEP under perfect foresight the consumption levels at any two adjacent dates are linked via the following "first-order conditions" (cf. equation (17)): $c_{t+1}^* \;=\; c_t^*\left(\beta\delta(1+r_{t+1}^*)\right)^{\frac{1}{\rho}}\frac{1+\Delta_t^T}{1+\beta^{\frac{1}{\rho}}\Delta_t^T}$. Since evidently, $1+\Delta_t^T > 1+\beta^{\frac{1}{\rho}}\Delta_t^T > \beta^{\frac{1}{\rho}} + \beta^{\frac{1}{\rho}}\Delta_t^T$, we have

$$\frac{c_{t+1}^*}{c_t^*} \;=\; \left(\beta\delta(1+r_{t+1}^*)\right)^{\frac{1}{\rho}}\frac{1+\Delta_t^T}{1+\beta^{\frac{1}{\rho}}\Delta_t^T} > \left(\beta\delta(1+r_{t+1}^*)\right)^{\frac{1}{\rho}}\,,$$

and

$$\frac{c_{t+1}^*}{c_t^*} \;=\; \left(\delta(1+r_{t+1}^*)\right)^{\frac{1}{\rho}}\frac{\beta^{\frac{1}{\rho}}+\beta^{\frac{1}{\rho}}\Delta_t^T}{1+\beta^{\frac{1}{\rho}}\Delta_t^T} < \left(\delta(1+r_{t+1}^*)\right)^{\frac{1}{\rho}}\,,$$

and hence (C.12) holds. ∎

**Step 3. Proof of the lemma**.

Now we can show that the sequence of capital stocks on a $T$-horizon SEP under perfect foresight is bounded from above by the sequence $\{\bar{k}_{t+1}(T)\}_{t=0}^T$.

**Claim C.5.** *For all $0 \le t \le T$, we have $k_{t+1}^* < \bar{k}_{t+1}$.*

*Proof.* Suppose the opposite, i.e., that while $k_0^* = \bar{k}_0$ and $k_\tau^* < \bar{k}_\tau$ (if $\tau > 0$), we have $k_{\tau+1}^* \ge \bar{k}_{\tau+1}$ for some $\tau \le T-1$. Then

$$c_\tau^* \;=\; f(k_\tau^*) - k_{\tau+1}^* \;\le\; f(\bar{k}_\tau) - k_{\tau+1}^* \;\le\; f(\bar{k}_\tau) - \bar{k}_{\tau+1} \;=\; \bar{c}_\tau. \tag{C.13}$$

Let us show that also $k_{\tau+2}^* \ge \bar{k}_{\tau+2}$. Suppose the opposite, i.e., $k_{\tau+2}^* < \bar{k}_{\tau+2}$. Then $c_{\tau+1}^* = f(k_{\tau+1}^*) - k_{\tau+2}^* \ge f(\bar{k}_{\tau+1}) - k_{\tau+2}^* > f(\bar{k}_{\tau+1}) - \bar{k}_{\tau+2} = \bar{c}_{\tau+1}$. At the same time, it follows from (C.12), (C.13) and (C.10) that

$$c^*_{\tau+1} < \left(\delta(1+r^*_{t+1})\right)^{\frac{1}{\rho}} c^*_\tau \leq (\delta(1+\bar{r}_{t+1}))^{\frac{1}{\rho}} c^*_\tau \leq (\delta(1+\bar{r}_{t+1}))^{\frac{1}{\rho}} \bar{c}_\tau = \bar{c}_{\tau+1}.$$

This contradiction shows that $k^*_{\tau+2} \geq \bar{k}_{\tau+2}$.

Repeating the argument, we obtain that if $k^*_{\tau+1} \geq \bar{k}_{\tau+1}$, then $k^*_{t+1} \geq \bar{k}_{t+1}$ for all $\tau+1 \leq t \leq T$. It then follows from (C.12) that $c^*_t \leq \bar{c}_t$ for all $\tau \leq t \leq T$.

Consider now the $T$-horizon date-$T$ consumer optimum starting from $k^*_T$ at given $1+r^*_{T+1}$ and $w^*_{T+1}$. Since $\beta < 1$, by the first-order condition and (C.10), $c^{**T}_{T+1} < \left(\delta(1+r^*_{T+1})\right)^{\frac{1}{\rho}} c^*_T \leq (\delta(1+\bar{r}_{t+1}))^{\frac{1}{\rho}} c^*_T \leq (\delta(1+\bar{r}_{t+1}))^{\frac{1}{\rho}} \bar{c}_T = \bar{c}_{T+1}$. However, it follows from the budget constraint and (C.11) that

$$\begin{aligned} c^{**T}_{T+1} &= (1+r^*_{T+1})\,(f(k^*_T) - c^*_T) + w^*_{T+1} = (1+r^*_{T+1})k^*_{T+1} + w^*_{T+1} \\ &= f(k^*_{T+1}) \geq f(\bar{k}_{T+1}) = (1+\bar{r}_{T+1})\left(f(\bar{k}_T) - \bar{c}_T\right) + \bar{w}_{T+1} = \bar{c}_{T+1}, \end{aligned}$$

and this contradiction shows that indeed $k^*_{t+1} < \bar{k}_{t+1}$ for all $0 \leq t \leq T$. ■

## C.4 Proof of Lemma 2.3

**Step 1. Upper and lower bounds for capital**.

Let $\{\underline{c}_t, \underline{k}_{t+1}\}_{t=0}^\infty$ be the $\beta\delta$-optimal path, i.e., the optimal path starting from $\underline{k}_0 = k^*_0$ in the standard Ramsey model with a discount factor $\beta\delta$. It is clear that $\underline{c}_t = \lim_{T\to\infty} \underline{c}_t(T)$ and $\underline{k}_{t+1} = \lim_{T\to\infty} \underline{k}_{t+1}(T)$.

Similarly, let $\{\bar{c}_t, \bar{k}_{t+1}\}_{t=0}^\infty$ be the $\delta$-optimal path, i.e., the optimal path starting from $\bar{k}_0 = k^*_0$ in the standard Ramsey model with a discount factor $\delta$. As above, it is clear that $\bar{c}_t = \lim_{T\to\infty} \bar{c}_t(T)$ and $\bar{k}_{t+1} = \lim_{T\to\infty} \bar{k}_{t+1}(T)$.

**Step 2. A closed-form expression for consumption**.

Now note that by construction of the sequence (14), $\{c^*_t, s^*_t, k^*_{t+1}, r^*_{t+1}, w^*_{t+1}\}_{t=0}^\infty$, we have $c^*_t = \lim_{T\to\infty} c^*_t(T)$, $s^*_t = \lim_{T\to\infty} s^*_t(T)$, $k^*_{t+1} = \lim_{T\to\infty} k^*_{t+1}(T)$, $r^*_{t+1} = \lim_{T\to\infty} r^*_{t+1}(T)$, and $w^*_{t+1} = \lim_{T\to\infty} w^*_{t+1}(T)$.

Taking the limit $T \to \infty$ in (13), we obtain that $\underline{k}_{t+1} \leq k^*_{t+1} \leq \bar{k}_{t+1}$ for all

$t \geq 0$. It immediately follows that $\underline{s}_t \leq s_t^* \leq \bar{s}_t$, $\underline{w}_t \leq w_t^* \leq \bar{w}_t$, and $\bar{r}_t \leq r_t^* \leq \underline{r}_t$.

Note also that $c_0^* = f(k_0^*) - k_1^* = f(\bar{k}_0) - k_1^* \geq f(\bar{k}_0) - \bar{k}_1 = \bar{c}_0$, and taking the limit $T \to \infty$ in (C.12), we get

$$\frac{c_{t+1}^*}{c_t^*} = \lim_{T\to\infty} \frac{c_{t+1}^*(T)}{c_t^*(T)} \geq (\beta\delta(1+r_{t+1}^*))^{\frac{1}{\rho}} \geq (\beta\delta(1+\bar{r}_{t+1}))^{\frac{1}{\rho}},$$

so that $c_t^* > 0$ for all $t \geq 0$.

Let us show that $c_\tau^*$ is given by a limit of (C.3) as $T \to \infty$.

**Claim C.6.** *For all $\tau \geq 0$, we have*

$$c_\tau^* = \frac{f(k_\tau^*) + \sum_{t=\tau+1}^{\infty} \frac{w_t^*}{(1+r_{\tau+1}^*)\cdots(1+r_t^*)}}{1 + (\beta\delta)^{\frac{1}{\rho}}(1+r_{\tau+1}^*)^{\frac{1-\rho}{\rho}} + \ldots + (\beta\delta^t)^{\frac{1}{\rho}}\left((1+r_{\tau+1}^*)\cdots(1+r_{\tau+t}^*)\right)^{\frac{1-\rho}{\rho}} + \ldots}. \tag{C.14}$$

*Proof.* Note that (C.3) can be written as

$$c_\tau^*(T)\left(1 + \sum_{t=\tau+1}^{T+1} (\beta\delta^{t-\tau})^{\frac{1}{\rho}}\left((1+r_{\tau+1}^*(T))\cdots(1+r_t^*(T))\right)^{\frac{1-\rho}{\rho}}\right) = f(k_\tau^*(T)) + \sum_{t=\tau+1}^{T+1} \frac{w_t^*(T)}{(1+r_{\tau+1}^*(T))\cdots(1+r_t^*(T))}. \tag{C.15}$$

By (13), the right hand side in (C.15) is bounded from above for any $T$, and hence there exists a finite limit as $T \to \infty$:

$$\lim_{T\to\infty}\left\{f(k_\tau^*(T)) + \sum_{t=\tau+1}^{T+1} \frac{w_t^*(T)}{(1+r_{\tau+1}^*(T))\cdots(1+r_t^*(T))}\right\} = f(k_\tau^*) + \sum_{t=\tau+1}^{\infty} \frac{w_t^*}{(1+r_{\tau+1}^*)\cdots(1+r_t^*)} \leq f(\bar{k}_\tau) + \sum_{t=\tau+1}^{\infty} \frac{\bar{w}_t}{(1+\bar{r}_{\tau+1})\cdots(1+\bar{r}_t)}.$$

It then follows from (C.15) that there exists a finite limit of its left hand side as $T \to \infty$, and hence the following series converges:

$$\begin{aligned}
&\lim_{T\to\infty}\left(1+\sum_{t=\tau+1}^{T+1}(\beta\delta^{t-\tau})^{\frac{1}{\rho}}\left((1+r^*_{\tau+1}(T))\cdots(1+r^*_t(T))\right)^{\frac{1-\rho}{\rho}}\right)\\
&\quad = 1+(\beta\delta)^{\frac{1}{\rho}}(1+r^*_{\tau+1})^{\frac{1-\rho}{\rho}}+\ldots+(\beta\delta^t)^{\frac{1}{\rho}}\left((1+r^*_{\tau+1})\cdots(1+r^*_{\tau+t})\right)^{\frac{1-\rho}{\rho}}+\ldots .
\end{aligned}$$

Therefore, taking the limit $T \to \infty$ in (C.15), we obtain (C.14). ■

**Step 3. Infinite horizon date-$\tau$ consumer optimum**.

For each $\tau \geq 0$, consider the sequence $\{c_t^{**\tau}\}_{t=\tau}^{\infty}$, which is defined as follows:

$$\begin{aligned}
c_\tau^{**\tau} &= c_\tau^*\,, \qquad\qquad c_{\tau+1}^{**\tau} = \left(\beta\delta(1+r^*_{\tau+1})\right)^{\frac{1}{\rho}}c_\tau^{**\tau}\,,\\
c_{t+1}^{**\tau} &= \left(\delta(1+r^*_{t+1})\right)^{\frac{1}{\rho}}c_t^{**\tau}, \qquad t \geq \tau+1\,.
\end{aligned} \tag{C.16}$$

**Claim C.7.** *The sequence $\{c_t^{**\tau}\}_{t=\tau}^{\infty}$ is the sequence of consumptions on the date-$\tau$ consumer optimum starting from $k_\tau^*$ at given $\{r^*_{t+1}\}_{t=\tau}^{\infty}$ and $\{w^*_{t+1}\}_{t=\tau}^{\infty}$.*

*Proof.* Let us show that $\{c_t^{**\tau}\}_{t=\tau}^{\infty}$ is the solution to the following problem:

$$\begin{aligned}
\max_{c_t\geq 0}\; u(c_\tau)+\beta\sum_{t=\tau+1}^{\infty}\delta^{t-\tau}u(c_t)\,, \qquad \text{s. t. } c_\tau+\sum_{t=\tau+1}^{\infty}\frac{c_t}{(1+r^*_{\tau+1})\cdots(1+r^*_t)}\\
\leq f(k^*_\tau)+\sum_{t=\tau+1}^{\infty}\frac{w^*_t}{(1+r^*_{\tau+1})\cdots(1+r^*_t)}.
\end{aligned}$$

It follows from (C.16) that $\{c_t^{**\tau}\}_{t=\tau}^{\infty}$ satisfies the first-order conditions for the solution to this problem. By combining (C.16) and (C.14), it is easily seen that

$$\begin{aligned}
c_\tau^{**\tau}+\sum_{t=\tau+1}^{\infty}\frac{c_t^{**\tau}}{(1+r^*_{\tau+1})\cdots(1+r^*_t)} = c^*_\tau\Big(1+(\beta\delta)^{\frac{1}{\rho}}(1+r^*_{\tau+1})^{\frac{1-\rho}{\rho}}+\ldots\\
+(\beta\delta^t)^{\frac{1}{\rho}}\left((1+r^*_{\tau+1})\cdots(1+r^*_{\tau+t})\right)^{\frac{1-\rho}{\rho}}+\ldots\Big) = f(k^*_\tau)+\sum_{t=\tau+1}^{\infty}\frac{w^*_t}{(1+r^*_{\tau+1})\cdots(1+r^*_t)},
\end{aligned}$$

so that $\{c_t^{**\tau}\}_{t=\tau}^{\infty}$ satisfies the budget constraint in this problem.

Consider the utility on the path $\{c_t^{**\tau}\}_{t=\tau}^{\infty}$, $U_\tau = u(c_\tau^{**\tau})+\beta\sum_{t=\tau+1}^{\infty}\delta^{t-\tau}u(c_t^{**\tau})$.

Using (C.16) and (C.14), we obtain that

$$U_\tau = \frac{(c_\tau^{**\tau})^{1-\rho}}{1-\rho} + \beta \sum_{t=\tau+1}^{\infty} \delta^{t-\tau} \frac{(c_t^{**\tau})^{1-\rho}}{1-\rho} = \frac{(c_\tau^{**\tau})^{1-\rho}}{1-\rho} \left( 1 + \beta \sum_{t=\tau+1}^{\infty} \delta^{t-\tau} \left( \frac{c_t^{**\tau}}{c_\tau^{**\tau}} \right)^{1-\rho} \right)$$

$$= \frac{1}{1-\rho} \frac{c_\tau^{**\tau}}{(c_\tau^{**\tau})^\rho} \left( 1 + \sum_{t=\tau+1}^{\infty} (\beta \delta^{t-\tau})^{\frac{1}{\rho}} \left( (1+r_{\tau+1}^*) \cdots (1+r_t^*) \right)^{\frac{1-\rho}{\rho}} \right)$$

$$= \frac{1}{1-\rho} \frac{1}{(c_\tau^*)^\rho} \left( f(k_\tau^*) + \sum_{t=\tau+1}^{\infty} \frac{w_t^*}{(1+r_{\tau+1}^*) \cdots (1+r_t^*)} \right).$$

It is now clear that $-\infty < U_\tau < \infty$, which completes the proof of the claim. ■

**Step 4. Ensuring existence**.

Claim C.7 shows that the sequence $\{c_t^{**\tau}\}_{t=\tau}^\infty$, determined by (C.16), is the sequence of consumptions on the date-$\tau$ consumer optimum starting from $k_\tau^*$ at given $\{r_{t+1}^*\}_{t=\tau}^\infty$ and $\{w_{t+1}^*\}_{t=\tau}^\infty$. Therefore, for each $\tau$, $c_\tau^*$ and $s_\tau^* = f(k_\tau^*) - c_\tau^*$ are the first elements of the date-$\tau$ consumer optimum starting from $k_\tau^*$ at given $\{r_{t+1}^*\}_{t=\tau}^\infty$ and $\{w_{t+1}^*\}_{t=\tau}^\infty$. Thus the sequence $\{c_t^*, s_t^*, k_t^*, r_t^*, w_t^*\}_{t=0}^\infty$, given by (14), is a sliding equilibrium path under perfect foresight starting from $s_{-1}^* = k_0^*$, which completes the proof of Lemma 2.3 and the proof of existence theorem as well.

# D Proof of Lemma 3.1

Let $c_\tau^{**\tau}$ be the date-$\tau$ consumption in the date-$\tau$ consumer optimum starting from $s_{\tau-1}^*$ at given $\{r_t^*\}_{t=\tau}^\infty$ and $\{w_t^*\}_{t=\tau}^\infty$. It follows from (5) that

$$c_\tau^{**\tau} = \frac{(1+r_\tau^*) s_{\tau-1}^* + w_\tau^* + \sum_{t=\tau+1}^\infty \frac{w_t^*}{(1+r_{\tau+1}^*)\cdots(1+r_t^*)}}{1 + (\beta\delta)^{\frac{1}{\rho}} (1+r_{\tau+1}^*)^{\frac{1-\rho}{\rho}} + \ldots + (\beta\delta^t)^{\frac{1}{\rho}} \left( (1+r_{\tau+1}^*) \cdots (1+r_{\tau+t}^*) \right)^{\frac{1-\rho}{\rho}} + \ldots}.$$

Due to the properties of $r^*$ (cf. Lemma 1.2), $c_\tau^{**\tau}$ is well-defined for all $\tau$:

$$\frac{w_{t+1}^*}{(1+r_{\tau+1}^*)\cdots(1+r_{t+1}^*)} \frac{(1+r_{\tau+1}^*)\cdots(1+r_t^*)}{w_t^*} = \frac{w_{t+1}^*}{w_t^*} \frac{1}{1+r_{t+1}^*} \xrightarrow[t\to\infty]{} \frac{1}{1+r^*} < 1,$$

$$\frac{(\beta\delta^{t+1})^{\frac{1}{\rho}} \left( (1+r_{\tau+1}^*)\cdots(1+r_{t+1}^*) \right)^{\frac{1-\rho}{\rho}}}{(\beta\delta^t)^{\frac{1}{\rho}} \left( (1+r_{\tau+1}^*)\cdots(1+r_t^*) \right)^{\frac{1-\rho}{\rho}}} = \delta^{\frac{1}{\rho}} (1+r_{t+1}^*)^{\frac{1-\rho}{\rho}} \xrightarrow[t\to\infty]{} \delta^{\frac{1}{\rho}} (1+r^*)^{\frac{1-\rho}{\rho}} < 1.$$

By the d'Alembert's ratio test both infinite series in the equation for $c_\tau^{**\tau}$ converge.

Let $\Delta_{t+1}$ be given by (15). By the same argument as above, $0 < \Delta_t < \infty$ for all $t$. Now, on a SEP under perfect foresight, we have for all $\tau \geq 0$,

$$c_\tau^* = c_\tau^{**\tau} = \frac{(1+r_\tau^*)s_{\tau-1}^* + w_\tau^* + \sum_{t=\tau+1}^{\infty} \frac{w_t^*}{(1+r_{\tau+1}^*)\cdots(1+r_t^*)}}{1+(\beta\delta)^{\frac{1}{\rho}}(1+r_{\tau+1}^*)^{\frac{1-\rho}{\rho}}(1+\Delta_{\tau+1})}, \tag{D.1}$$

$$k_{\tau+1}^* = s_\tau^* = (1+r_\tau^*)s_{\tau-1}^* + w_\tau^* - c_\tau^*, \quad 1+r_\tau^* = f'(k_\tau^*), \quad w_\tau^* = f(k_\tau^*) - f'(k_\tau^*)k_\tau^*.$$

Consider the value $c_{\tau+1}^*(1+\beta^{\frac{1}{\rho}}\Delta_{\tau+1})$. By (16) and (D.1), we have

$$\begin{aligned} &c_{\tau+1}^*\left(1+\beta^{\frac{1}{\rho}}\Delta_{\tau+1}\right) = c_{\tau+1}^*\left(1+(\beta\delta)^{\frac{1}{\rho}}(1+r_{\tau+2}^*)^{\frac{1-\rho}{\rho}}(1+\Delta_{\tau+2})\right) \\ &= (1+r_{\tau+1}^*)\left(s_\tau^* + \sum_{t=\tau+1}^{\infty}\frac{w_t^*}{(1+r_{\tau+1}^*)\cdots(1+r_t^*)}\right) = (1+r_{\tau+1}^*)\left(s_\tau^* - (1+r_\tau^*)s_{\tau-1}^*\right. \\ &\left. - w_\tau^* + c_\tau^* + c_\tau^*(\beta\delta)^{\frac{1}{\rho}}(1+r_{\tau+1}^*)^{\frac{1-\rho}{\rho}}(1+\Delta_{\tau+1})\right) = c_\tau^*(\beta\delta)^{\frac{1}{\rho}}(1+r_{\tau+1}^*)^{\frac{1}{\rho}}(1+\Delta_{\tau+1}). \end{aligned}$$

Thus, on a SEP under perfect foresight the consumption levels at two adjacent dates are linked via the "first-order conditions" (17), which proves the lemma.

# E Proof of Theorem 4

The proof is by contradiction. Suppose that $k^* \leq k^\circ$, so $r^\circ \leq r^*$ and $w^\circ \geq w^*$. Let $c(r^\circ, w^\circ)$ be the date-$\tau$ consumption in the date-$\tau$ consumer optimum starting from $s^\circ$ at given constant $r^\circ$ and $w^\circ$. The following lemma shows that $c(r^\circ, w^\circ) \geq c^\circ$.

**Lemma E.1.** *Suppose that* $0 < \rho \leq 1$, *and* $k^* \leq k^\circ$. *Then* $c(r^\circ, w^\circ) \geq c^\circ$.

*Proof.* Since $r^\circ \leq r^*$ and $0 < \rho \leq 1$, we have $\delta(1+r^\circ)^{1-\rho} \leq \delta(1+r^*)^{1-\rho} < 1$. It then follows from Lemma 1.1 that $c(r^\circ, w^\circ)$ exists and is given by

$$c(r^\circ, w^\circ) = \frac{1-\delta^{\frac{1}{\rho}}(1+r^\circ)^{\frac{1-\rho}{\rho}}}{1-\delta^{\frac{1}{\rho}}(1+r^\circ)^{\frac{1-\rho}{\rho}} + (\beta\delta)^{\frac{1}{\rho}}(1+r^\circ)^{\frac{1-\rho}{\rho}}} \cdot \frac{1+r^\circ}{r^\circ} \cdot (r^\circ s^\circ + w^\circ).$$

Taking into account that $c^\circ = (1+r^\circ)s^\circ + w^\circ - s^\circ = r^\circ s^\circ + w^\circ$, we get

$c(r^\circ, w^\circ) = Z(r^\circ)c^\circ$, where the function $Z(r)$ is defined as

$$Z(r) \;=\; \frac{1-\delta^{\frac{1}{\rho}}(1+r)^{\frac{1-\rho}{\rho}}}{1-\delta^{\frac{1}{\rho}}(1+r)^{\frac{1-\rho}{\rho}}+(\beta\delta)^{\frac{1}{\rho}}(1+r)^{\frac{1-\rho}{\rho}}}\cdot\frac{1+r}{r}.$$

It follows from the proof of Lemma 1.1 that $Z(r^*) = 1$, where $r^*$ is the interest rate on a SSE under perfect foresight.

Let us show that for all $r \le r^*$, $Z(r) \ge Z(r^*) = 1$. Repeating the argument used in the proof of Theorem 1, we find that $Z(r) \ge 1$ for $\frac{1}{1+r} \;\le\; \frac{\left(\frac{1}{1+r}\right)^{\frac{1}{\rho}}-(\beta\delta)^{\frac{1}{\rho}}}{\delta^{\frac{1}{\rho}}-(\beta\delta)^{\frac{1}{\rho}}}$.

Let $\gamma = \frac{1}{1+r}$, and consider the functions $L(\gamma) = \gamma$ and $R(\gamma) = \frac{\gamma^{\frac{1}{\rho}}-(\beta\delta)^{\frac{1}{\rho}}}{\delta^{\frac{1}{\rho}}-(\beta\delta)^{\frac{1}{\rho}}}$ (cf. proof of Lemma 1.2). Since they are monotone and $R(\gamma^*) = L(\gamma^*)$, while $R(\delta) = 1 > \delta = L(\delta)$, it follows that $R(\gamma) \ge L(\gamma)$ for all $\gamma$ such that $\gamma^* \le \gamma < \delta$. In terms of interest rates, $Z(r) \ge 1$ for $r \le r^*$. Since by assumption, $r^\circ \le r^*$, $Z(r^\circ) \ge Z(r^*) = 1$. Thus, $c(r^\circ, w^\circ) = Z(r^\circ)c^\circ \ge c^\circ$, which proves the lemma. ■

However, the following lemma shows that the opposite inequality holds.

**Lemma E.2.** *Suppose that* $0 < \rho \le 1$, *and* $k^* \le k^\circ$. *Then* $c^\circ \;>\; c(r^\circ, w^\circ)$.

*Proof.* Recall that $c^\circ$ is the date-$\tau$ consumption in the date-$\tau$ consumer optimum starting from $s^\circ$ at given $\{r_t^{*\tau}\}_{t=\tau}^\infty$ and $\{w_t^{*\tau}\}_{t=\tau}^\infty$. It follows from (5) that

$$c^\circ = \frac{(1+r^\circ)s^\circ + w^\circ + \frac{w^\circ}{1+r^\circ} + \sum_{t=\tau+2}^\infty \frac{w_t^{*\tau}}{(1+r^\circ)\cdots(1+r_t^{*\tau})}}{1+(\beta\delta)^{\frac{1}{\rho}}(1+r^\circ)^{\frac{1-\rho}{\rho}}+\ldots+(\beta\delta^t)^{\frac{1}{\rho}}\left((1+r^\circ)\cdots(1+r_{\tau+t}^{*\tau})\right)^{\frac{1-\rho}{\rho}}+\ldots}. \qquad \text{(E.1)}$$

Let us check that $c^\circ$ is well-defined. Since the truncation of the date-$\tau$ optimal path which starts at date $\tau+1$ is the $\delta$-optimal path, $r_t^{*\tau}$ and $w_t^{*\tau}$ converge to the corresponding modified golden rule levels for the discount factor $\delta$. Thus,

$$\frac{w_{t+1}^{*\tau}}{(1+r^\circ)\cdots(1+r_{t+1}^{*\tau})}\,\frac{(1+r^\circ)\cdots(1+r_t^{*\tau})}{w_t^*} = \frac{w_{t+1}^{*\tau}}{w_t^{*\tau}}\,\frac{1}{1+r_{t+1}^{*\tau}} \xrightarrow[t\to\infty]{} \delta < 1,$$

$$\frac{(\beta\delta^{t+1})^{\frac{1}{\rho}}\left((1+r^\circ)\cdots(1+r_{t+1}^{*\tau})\right)^{\frac{1-\rho}{\rho}}}{(\beta\delta^t)^{\frac{1}{\rho}}\left((1+r^\circ)\cdots(1+r_t^{*\tau})\right)^{\frac{1-\rho}{\rho}}} = \delta^{\frac{1}{\rho}}(1+r_{t+1}^{*\tau})^{\frac{1-\rho}{\rho}} \xrightarrow[t\to\infty]{} \delta^{\frac{1}{\rho}}\delta^{\frac{\rho-1}{\rho}} = \delta < 1.$$

By the d'Alembert's ratio test both infinite series in (E.1) converge.

Let us compare $c^\circ$ and $c(r^\circ, w^\circ)$. Since $\{r_t^{*\tau}\}_{t=\tau+2}^{\infty}$ is the sequence of interest rates on the $\delta$-equilibrium path starting from $k^\circ < k^\delta$, it is decreasing, and $r_{t+2}^{*\tau} < r^\circ$. Similarly, the sequence $\{w_t^{*\tau}\}_{t=\tau+2}^{\infty}$ is increasing, and $w_{t+2}^{*\tau} > w^\circ$.

It is evident from (E.1) that $c^\circ$ is increasing in $w_t^{*\tau}$ for all $t \geq \tau + 2$. The numerator in (E.1) is decreasing in $r_t^{*\tau}$, and for $0 < \rho \leq 1$, the denominator in (E.1) is non-decreasing in $r_t^{*\tau}$. Hence when $0 < \rho \leq 1$, $c^\circ$ is decreasing in $r_t^{*\tau}$ for any $t \geq \tau + 2$. Therefore, $c^\circ > c(r^\circ, w^\circ)$, which proves the lemma. ■

Since Lemmas E.1 and E.2 contradict each other, $k^* > k^\circ$ for $0 < \rho \leq 1$, and hence $s^* > s^\circ$, $r^* < r^\circ$, $w^* > w^\circ$, and $c^* = f(k^*) - k^* > f(k^\circ) - k^\circ = c^\circ$.


# Acknowledgements

We thank Hossein Farzin, Alexia Fürnkranz-Prskawetz, Emanuel Gasteiger, Michael Greinecker, Franz X. Hof, and Rick van der Ploeg for useful discussions and suggestions. Earlier versions of the paper greatly benefited from comments of participants of the Public Economic Theory (PET) 2019 Annual Conference (Strasbourg, France); 5th International Workshop on Economic Growth, Environment and Natural Resources (St. Petersburg, Russia, 2019); Economics Research Seminar at Vienna University of Technology (2019); IVth Russian Economic Congress (2020).


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