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Compromise in combinatorial vote

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Abstract We consider collective choice problems where the set of social outcomes is a Cartesian product of finitely many finite sets. We assume that each individual is assigned a two-level preference, defined as a pair involving a vector of strict rankings of elements in each of the sets and a strict ranking of social outcomes. A voting rule is called (resp. weakly) product stable at some two-level preference profile if every (resp. at least one) outcome formed by separate coordinate-wise choices is also an outcome of the rule applied to preferences over social outcomes. We investigate the (weak) product stability for the specific class of compromise solutions involving $q-$approval rules, where q lies between 1 and the number I of voters. Given a finite set $\mathcal{X}$ and a profile of I linear orders over $\mathcal{X}$, a $q-$approval rule selects elements of $\mathcal{X}$ that gather the largest support above q at the highest rank in the profile. Well-known $q-$approval rules are the Fallback Bargaining solution ($q = I$) and the Majoritarian Compromise ($q = \lceil \frac{I}{2} \rceil$). We assume that coordinate-wise rankings and rankings of social outcomes are related in a neutral way, and we investigate the existence of neutral two-level preference domains that ensure the weak product stability of $q-$approval rules. We show that no such domain exists unless either $q = I$ or very special cases prevail. Moreover, we characterize the neutral two-level preference domains over which the Fallback Bargaining solution is weakly product stable.

Key words Compromise – Multidimensional voting – Consistency – Product stability

1 Introduction

In a large variety of real-life collective choice problems, social outcomes are defined as elements of the Cartesian product of finite sets. Multiple referendum, elections of designated-post committees of representatives, or the choice by a group of collaborators of the agenda involving a series of weekly meetings are examples of such problems. In a multiple referendum, a society has to choose between accepting or rejecting each of finitely many proposals.[1] A fixed number of seats, or positions, have to be filled in a designated-post committee election, and the member selected for each seat is chosen from a finite set of candidates competing for that seat. If weekly meetings are logically ordered in time, choosing an agenda can also be formalized as the choice of a designated post committee, where dates and meetings are respectively interpreted as candidates and seats. Social outcomes in all these examples can be formalized as a vector in $\mathcal{A}_1 \times \ldots \times \mathcal{A}_M$, where $\mathcal{A}_1, \ldots, \mathcal{A}_M$ are finite sets of (coordinate-wise) alternatives.[2] As the number of social outcomes exponentially grows with the number of coordinates, it is hardly possible in practice that individuals report their preference about outcomes. To overcome this difficulty, a commonly used procedure consists of making separate coordinate-wise choices and aggregating these choices into a social outcome. However, such a procedure may have severe drawbacks. In particular, multiple election paradoxes highlight the fact that voting separately on each of finitely many issues may result in an obviously undesirable outcome. A reason usually put ahead is that making separate coordinate-wise choices ignores the potential existence of preference dependencies between coordinates.[3] However, problems arise even in the absence of preference dependencies. Indeed, Özkal-Sanver and Sanver (2006) show that in multiple referendum, choosing issue-wise (i.e., over each coordinate separately) according to the majority will may yield a Pareto dominated outcome even when voters' preferences over outcomes are separable. Benoît and Kornhauser (2010) strengthen this result by proving that under separable preferences over outcomes, if there are at least three issues (or two issues with at least three alternatives for one of them), dictatorship is the only issue-wise voting procedure that always gives a Pareto optimal outcome. As an immediate consequence, a Pareto efficient voting rule applied coordinate-wise may not yield an outcome consistent with the one that would arise if that voting rule was applied to preferences over outcomes. More generally, selecting alternatives coordinate-wise (sequential procedure) may give an outcome different from the one arising from selecting all alternatives at once (direct procedure).

Compound-majority paradoxes are well-known instances of this fact. In particular, the Ostrogorski paradox considers multiple referendum where each voter is characterized by an ideal outcome and have separable preferences over outcomes based on the Hamming distance to this ideal.[4] Under these assumptions, the outcome formed by coordinate-wise majority winners in a multiple referendum may be defeated by another outcome according to simple majority voting (Daudt and Rae, 1976; Bezembinder and van Acker, 1985; Deb and Kelsey, 1987; Laffond and Lainé, 2006). Actually, as shown by Hollard and Le Breton (1996), every majority preference over outcomes can be generated from some profile of separable preferences over outcomes. This result is generalized by Vidu (1999,2002) to the case where there are more than two alternatives per coordinate. Indeed, the majority preference over outcomes may involve any set of cycles at some profile of separable and seat-wise single-peaked preferences over outcomes.

A natural question is whether one can identify restrictions upon preferences for which sequential and direct procedures yield mutually consistent outcomes. More precisely, pick a voting rule F that operates for a variable number of alternatives. Moreover, assume that each voter is characterized by a pair of coordinate-wise and outcome-wise preferences. Call profile a pair $(\mathbf{P}, \mathbf{p})$ formed by a preference profile

[1] The citizens of California are familiar with the organization of multiple referendum. For instance, 12 ballot measures were certified to appear on the ballot for the election on November 3, 2020. Issues at stake covered a vast set of different topics, such as restoring the right to vote to people convicted of felonies who are on parole, expanding local governments' power to use rent control, or changing tax assessment transfers and inheritance rules. A debate is currently held in France about the pros and cons of instituting multiple referendum.

[2] A multiple referendum corresponds to the special case where all sets $\mathcal{A}_1, \ldots, \mathcal{A}_M$ contain only two alternatives.

[3] The reader may refer to Lang and Xia (2016) for a review of voting methods that overcome the difficulty created by non-separable preferences.

[4] The Hamming distance between two outcomes is the number of coordinates they disagree upon.

over outcomes $\mathbf{P}$ and a profile of coordinate-wise preferences $\mathbf{p}$. Denote by $\mathbf{p}_m$ the preference profile over alternatives in coordinate m in $\{1,\ldots,M\}$. We say that F is (resp. weakly) product stable at profile $(\mathbf{P},\mathbf{p})$ if every (resp. at least one) sequential outcome in $F(\mathbf{p}_1)\times\ldots F(\mathbf{p}_M)$ belongs to the set of direct outcomes $F(\mathbf{P})$.

In this paper, we focus on a specific class of voting rules comprising $q-$approval rules, and we investigate the existence of restrictions upon profiles $(\mathbf{P},\mathbf{p})$ at which $q-$approval rules are (weakly) product stable. A $q-$approval rule operates as follows. Take any preference profile $\boldsymbol{\pi}$ of I linear orders over a finite set of alternatives, and take $q\in\{1,\ldots,I\}$. Then, pull down the stick in $\boldsymbol{\pi}$ from the top until at least one alternative is supported by at least q voters. In case of a tie, select all alternatives that gather the highest support. The $I-$approval rule is known as Fallback Bargaining (Brams and Kilgour, 2001), while the $\lceil\frac{I}{2}\rceil-$approval rule is known as the Majoritarian Compromise (Hurwicz and Sertel, 1999; Sertel and Yilmaz, 1999; Núñez and Sanver, 2021).

Many situations involve parties (individuals or states) bargaining over multiple issues. A commonly shared idea is that parties face trade-offs between issues in such situations and engage in complex strategies of vote trading, where losing satisfaction on some issues may be more than compensated by getting satisfaction on others. Which moral stigma is actually carried by the practice of vote trading, and whether it brings a benefit or a loss in terms of welfare is largely debated (for instance, see Casella and Macé, 2021, and references quoted there). Situations where a $q-$approval rule is (weakly) product stable are precisely those where bargaining issue by issue results in an outcome achievable when bargaining all-at-once. Hence, as the sequential bargaining outcome remains after bundling all coordinates in a single bargaining, there is no implicit possibility of trading votes under sincere voting behavior.

We define a preference as a pair comprising one vector of (individual) coordinate-wise linear orders and one linear order over outcomes. A (preference) domain is a subset of preferences, i.e., a subset of two-level preferences that are mutually related in some restricted way. Attention is restricted to neutral domains, where neutrality essentially means that the names of alternatives do not matter in the way the two preference levels are linked. We show that every neutral domain is isomorphic to a set of linear orders over rank-vectors, defined as elements of $\prod_{m\in\{1,\ldots,M\}}\{1,\ldots,|\mathcal{A}_m|\}$.

Our main conclusion is that for any value of q between 1 and I, where I stands for the number of voters, a $q-$approval rule fails product stability over every neutral domain, unless very special conditions prevail on the number of coordinates and number of voters. Moreover, if $q<I$, a similar negative result holds for weak product stability. However, provided a large enough number of voters, the Fallback Bargaining rule is weakly product stable if and only if all voters have a preference over outcomes that is lexicographic (with respect to coordinate-wise rankings) according to the same ordering of the coordinates. We conjecture that the lower bound we provide for the number of voters is tight.

This paper relates in spirit to a recent study by Aslan et al. (2021), who characterize in a similar setting preference domains for which bundling coordinate-wise Condorcet winners gives the Condorcet winner among outcomes. Surprisingly enough, it turns out that, again under a neutrality assumption, lexicographic preferences with respect to a common priority ordering of coordinates are necessary and sufficient for this property as they are for the weak product stability of the Fallback Bargaining rule.

The sequel is organized as follows. Section 2 is devoted to preliminaries. Product stability and weak product stability are defined in section 3. Neutral domains are formalized in section 4, and we formally define $q-$approval rules in section 5. All results are gathered in section 6, where we first consider the Fallback Bargaining rule in section 6.1, then all other $q-$approval rules in section 6.2. The paper ends with further comments on open research questions.

2 Preliminaries

Given any finite set $\mathcal{X} = \{z_1, \ldots, z_X\}$, $\mathcal{L}_\mathcal{X}$ stands for the set of linear orders over $\mathcal{X}$. For any $\pi \in \mathcal{L}_\mathcal{X}$, we write $\pi = [z_1 z_2 \ldots z_X]$ if $z_1 \; \pi \; z_2 \; \pi \ldots \pi \; z_X$. Moreover, for any $x \in \mathcal{X}$, the rank of x in π is defined by $\pi(x) = |\{y \in \mathcal{X} : y \; \pi \; x\}| + 1$.

We consider a set of voters $\mathcal{I} = \{1, \ldots, I\}$ with $I \geq 2$ confronting $M \geq 2$ mutually disjoint sets $\mathcal{A}_1, \ldots, \mathcal{A}_M$ with respective cardinalities $A_1, \ldots, A_M$. We denote $\{1, \ldots, M\}$ by $\mathcal{M}$, and write $\mathbb{L} = \prod_{m \in \mathcal{M}} \mathcal{L}_{\mathcal{A}_m}$ for the set of vector of linear orders, one for each coordinate $m \in \mathcal{M}$.

We assume that $A_m \geq 3$ for all $m \in \mathcal{M}$. A social outcome is an element $\overrightarrow{a}$ of $\mathcal{A} = \prod_{m \in \mathcal{M}} \mathcal{A}_m$. We usually write $\overrightarrow{a}$, $\overrightarrow{b}$, ... for outcomes such as $\overrightarrow{a} = (a_1, \ldots, a_M)$, $\overrightarrow{b} = (b_1, \ldots, b_M)$, ... For any $\overrightarrow{a} \in \mathcal{A}$ and any $m \in \mathcal{M}$, we write $(b_m, \overrightarrow{a}_{-m})$ for the outcome obtained from $\overrightarrow{a}$ by replacing a_m with b_m. Moreover, given another social outcome $\overrightarrow{b} \in \mathcal{A}$ and given any $\mathcal{M}' \subset \mathcal{M}$, we write $(\overrightarrow{a}_{\mathcal{M}'}, \overrightarrow{b}_{-\mathcal{M}'})$ for the outcome $\overrightarrow{c} = (c_1, \ldots, c_M)$ defined by $\forall m \in \mathcal{M}$, $c_m = \begin{cases} a_m & \text{if } m \in \mathcal{M}' \\ b_m & \text{if } m \in \mathcal{M} \setminus \mathcal{M}' \end{cases}$. Furthermore, for any $m^* \in \mathcal{M}$, we write $(\overrightarrow{a}_{<m^*}, \overrightarrow{b}_{\geq m^*})$ to denote the social outcome $(a_1, \ldots, a_{m^*-1}, b_{m^*}, \ldots, b_M)$, and similarly, $(\overrightarrow{a}_{\leq m^*}, \overrightarrow{b}_{>m^*})$ to denote the social outcome $(a_1, \ldots, a_{m^*}, b_{m^*+1}, \ldots, b_M)$.

We define the set of rank vectors $\mathcal{R} = \prod_{m \in \mathcal{M}} \{1, \ldots, A_m\}$. We usually write R, R', ... for rank vectors such as $R = (r_1, \ldots, r_M)$, $R' = (r'_1, \ldots, r'_M)$, ... Given an outcome $\overrightarrow{a} \in \mathcal{A}$ together with a coordinate preference p, the rank vector of $\overrightarrow{a}$ at p is the element of $\mathcal{R}$ defined by $R(\overrightarrow{a}, p) = (p_1(a_1), \ldots, p_M(a_M))$. Hence, if $M = 3$, $R(\overrightarrow{a}, p) = (2, 1, 4)$ means that $\overrightarrow{a}$ comprises the second-best alternative for p at coordinate 1, the top-alternative for p in coordinate 2, and the fourth-best one at p in coordinate 3.

Each voter $i \in \mathcal{I}$ is assigned to a two-level preference, defined as a pair $(P^i, p^i) \in \mathcal{L}_\mathcal{A} \times \mathbb{L}$. We call P^i the outcome preference of voter i, and we define outcome profiles as elements $\mathbf{P} = (P^i)_{i \in \mathcal{I}}$ of $(\mathcal{L}_\mathcal{A})^I$. Moreover, we call $m-$preference of voter i an element p^i_m of $\mathcal{L}_{\mathcal{A}_m}$, and we call coordinate preference of i an element $p^i = (p^i_1, \ldots, p^i_M)$ of $\mathbb{L}$. For any coordinate $m \in \mathcal{M}$, an $m-$profile is an element $\mathbf{p}_m = (p^i_m)_{i \in \mathcal{I}}$ of $(\mathcal{L}_{\mathcal{A}_m})^I$, and a coordinate profile is an element $\mathbf{p} = (p^i)_{i \in \mathcal{I}}$ of $(\mathbb{L})^I$. We call two-level profile a couple $(\mathbf{P}, \mathbf{p}) \in (\mathcal{L}_\mathcal{A})^I \times (\mathbb{L})^I$.

Two-level preferences can be interpreted in two different ways, each being related to which level of preference is taken as the premise. According to the first way, each voter has ex-ante a well-defined outcome preference, while some elicitation procedure allows to generate coordinate-wise rankings of alternatives from outcome preferences. For instance, one may ask each voter i with outcome preference P^i to rank all outcomes $(b_m, \overrightarrow{a}^i_{-m})$ where $b_m \in \mathcal{A}_m$ and $\overrightarrow{a}^i$ is the first-best outcome at P^i. This ranking provides the $m-$preference p^i_m by $\forall b_m, c_m \in \mathcal{A}_m$, $b_m \; p^i_m \; c_m$ if and only if $(b_m, \overrightarrow{a}^i_{-m}) \; P^i \; (c_m, \overrightarrow{a}^i_{-m})$. Doing so at each coordinate generates a unique coordinate preference from the outcome preference. Observe that taking P^i as premise and generating p^i from P^i does not require the separability of P^i. However, there is a non-ambiguous way to relate coordinate and outcome preferences whenever the latter is separable.[5] Indeed, the separability of P^i generates a (necessarily unique) coordinate preference $p^i = (p^i_m)_{m \in \mathcal{M}}$ defined by $\forall m \in \mathcal{M}$, $\forall b_m, c_m \in \mathcal{A}_m$, $b_m \; p^i_m \; c_m$ if and only if $(b_m, \overrightarrow{a}_{-m}) \; P^i \; (c_m, \overrightarrow{a}_{-m})$ at some $\overrightarrow{a} \in \mathcal{A}$. In contrast with this interpretation, we can take the coordinate preference p^i as premise, while outcome preferences are underlying. This prevails at instance in the case where the outcome is obtained by separate and simultaneous coordinate-wise choices.[6] Under this interpretation, outcome preferences are not announced,

[5] An outcome preference $P^i \in \mathcal{L}_\mathcal{A}$ is separable if $\forall \overrightarrow{a}, \overrightarrow{b} \in \mathcal{A}$, and $\forall m \in \mathcal{M}$, $\forall c_m, d_m \in \mathcal{A}_m$, we have $(c_m, \overrightarrow{a}_{-m}) \; P^i \; (d_m, \overrightarrow{a}_{-m})$ if and only if $(c_m, \overrightarrow{b}_{-m}) \; P^i \; (d_m, \overrightarrow{b}_{-m})$.

[6] If outcomes (resp. coordinates) are interpreted as committees (resp. designated seats), this procedure amounts to simultaneously selecting candidates seat by seat from voters' rankings of candidates running for that seat.

and therefore some assumption has to be retained about how coordinate preferences can be extended to outcome preferences.[7] Similar to the first interpretation, outcome preferences may fail separability. For instance, if outcomes have to be selected at once, voters having previously reported coordinate-wise rankings must rank outcomes. Due to potential complementary between alternatives, this ranking may be nonseparable w.r.t. announced coordinate-wise rankings.

3 Product stability of voting rules

A voting rule aggregates voters' preferences into one or several outcomes. Formally, at any finite set $\mathcal{X}$, a voting rule is a function $F : (\mathcal{L}_{\mathcal{X}})^I \to 2^{\mathcal{X}} \setminus \{\emptyset\}$. Observe that we define voting rules at alternative sets with arbitrary cardinality. Since a two-level preference structure prevails, a voting rule may be applied either to each $m-$profile or to the outcome profile. Applying a voting rule F to each $m-$profile $\mathbf{p}_m$ gives as social outcomes all vectors in $\prod_{m\in\mathcal{M}} F(\mathbf{p}_m)$ while applying F to outcome profile $\mathbf{P}$ gives $F(\mathbf{P})$ as the set of social outcomes. We refer to social outcomes in $\prod_{m\in\mathcal{M}} F(\mathbf{p}_m)$ as coordinate-wise outcomes (hereafter CWO) and social outcomes in $F(\mathbf{P})$ as full information outcomes (hereafter FIO).

We consider two ways of defining consistency between CWOs and FIOs.

Definition 1 *A voting rule F is (resp. weakly) product stable at the two-level profile $(\mathbf{P},\mathbf{p})$ if $\prod_{m\in\mathcal{M}} F(\mathbf{p}_m) \subseteq F(\mathbf{P})$ (resp. $[\prod_{m\in\mathcal{M}} F(\mathbf{p}_m)] \cap F(\mathbf{P}) \neq \emptyset$). Moreover, F is (resp. weakly) product stable over the domain $\mathcal{D} \subseteq \mathcal{L}_{\mathcal{A}} \times \mathbb{L}$ if F is (resp. weakly) product stable at every $(\mathbf{P},\mathbf{p}) \in \mathcal{D}^I$.*

With a product stable voting rule, choosing coordinate-wise according to some voting rule gives an outcome that would also be chosen from the knowledge of full preferences over social outcomes. In short, every CWO is an FIO. Hence, product stability means that every choice made from partial information about outcome preferences would be confirmed if the cost of full information was incurred. However, observe that some FIOs may not be achievable as CWOs. As weak product stability states that at least one CWO is an FIO, it should be seen as a minimal consistency requirement. However, since some CWOs may not be FIOs, and since those CWOs cannot be identified without knowing outcome preferences, weak product stability should be seen as necessary but not sufficient for a coordinate-wise procedure to be satisfactory.

4 Product stability over neutral domains

We define a domain as a non-empty subset $\mathcal{D}$ of $\mathcal{L}_{\mathcal{A}} \times \mathbb{L}$. Each domain generates a set of admissible two-level profiles $\mathcal{D}^I$, where each voter i is assigned to $(P^i, p^i) \in \mathcal{D}$. Attention is restricted to neutral domains.

Another possibility is iterative voting, where seat-wise choices are made in some order, the choice for one seat being publicly known before the choice is made for the next seat (see Meir, 2017 for a review of iterative voting).

[7] Studies involving two different preference levels usually involve the use of a preference extension. For a rich review of preference extensions that link preferences over alternatives to preferences over sets of alternatives, the reader may refer to Barberà et al. (2004). The importance of preference extensions is pointed out in many studies investigating the consistency between alternative-based and set-based collective choice (see Kaymak and Sanver, 2003; Kamwa and Merlin, 2018). In the case where outcomes are rankings rather than sets of alternatives, preference extensions play a critical role for the incentive compatibility of Arrovian aggregation (Bossert and Storcken, 1992; Bossert and Sprumont, 2014; Athanasoglu, 2016). In their analysis of product stability for Condorcet rules, in a framework similar to the present one, where outcomes are designated-post committees, Aslan et al. (2021) also use the concept of preference extension, which maps rankings of seat-wise candidates to rankings of committees.

Neutrality means the labeling of the alternatives should play no role in how social outcomes are ranked and is formally defined as follows. Pick any permutation vector $\sigma = (\sigma_1, \ldots, \sigma_M)$ where for each $m \in \mathcal{M}$, σ_m is a permutation of $\mathcal{A}_m$. For any $\overrightarrow{a} \in \mathcal{A}$, $\sigma(\overrightarrow{a}) = [\sigma_1(a_1), \ldots, \sigma_M(a_M)]$ stands for the committee obtained from $\overrightarrow{a}$ by operating all coordinate-wise permutations. Moreover, given any $P \in \mathcal{L}_\mathcal{A}$, we define $P_\sigma \in \mathcal{L}_\mathcal{A}$ by $\forall \overrightarrow{a}, \overrightarrow{b} \in \mathcal{A}$, $\overrightarrow{a}\ P\ \overrightarrow{b} \Leftrightarrow \sigma(\overrightarrow{a})\ P_\sigma\ \sigma(\overrightarrow{b})$. Similarly, given any $m \in \mathcal{M}$ and $p_m \in \mathcal{L}_{\mathcal{A}_m}$, we define $p_{m,\sigma_m} \in \mathcal{L}_{\mathcal{A}_m}$ by $\forall a_m, b_m \in \mathcal{A}_m$, $a_m\ p_m\ b_m \Leftrightarrow \sigma_m(a_m)\ p_{m,\sigma_m}\ \sigma_m(b_m)$. Moreover, we define $p_\sigma = (p_{m,\sigma_m})_{m\in\mathcal{M}}$.

Definition 2 *A domain $\mathcal{D}$ is neutral if and only if for any permutation vector $\sigma = (\sigma_m)_{m\in\mathcal{M}}$, for any two-level preference $(P,p) \in \mathcal{L}_\mathcal{A} \times \mathbb{L}$, $(P,p) \in \mathcal{D} \iff (P_\sigma, p_\sigma) \in \mathcal{D}$.*

Under neutrality, each $(P,p) \in \mathcal{D}$ generates the set $\Gamma(P,p) = \cup_{\sigma\in\Sigma}\{(P_\sigma, p_\sigma)\} \subseteq \mathcal{D}$, where Σ is the set of all permutation vectors. Obviously, the cardinality of each set $\Gamma(P,p)$ is equal to $\prod_{m\in\mathcal{M}} A_m!$. Hence, we can write a neutral domain $\mathcal{D}$ as a collection of sets $\Gamma(P,p)$. It turns out that one can span every neutral domain $\mathcal{D}$ by picking an arbitrary coordinate preference p and computing the union of all sets $\Gamma(P,p)$ where P is an outcome preference for which $(P,p) \in \mathcal{D}$.

Lemma 1 *If $\mathcal{D}$ is a neutral domain, then $\mathcal{D} = \cup_{(P,p)\in\mathcal{D}}\Gamma(P,p)$, where p is arbitrarily chosen in $\mathbb{L}$.*

Proof Pick an arbitrary $p \in \mathbb{L}$. By neutrality of $\mathcal{D}$, for any $P \in \mathcal{L}_\mathcal{A}$ with $(P,p) \in \mathcal{D}$, one has $\Gamma(P,p) \subseteq \mathcal{D}$. Thus $\forall p \in \mathbb{L}$, $\cup_{(P,p)\in\mathcal{D}}\Gamma(P,p) \subseteq \mathcal{D}$.

Now, pick any $p' \in \mathbb{L}$ and $P' \in \mathcal{L}_\mathcal{A}$ with $(P',p') \in \mathcal{D}$. We want to show that $(P',p') \in \cup_{(P,p)\in\mathcal{D}}\Gamma(P,p)$ for an arbitrary $p \in \mathbb{L}$. Clearly, if $p' = p$, then $(P',p') \in \cup_{(P,p)\in\mathcal{D}}\Gamma(P,p)$ is trivially satisfied. If $p' \neq p$, there exists $\sigma \in \Sigma$ such that $p' = p_\sigma$. Hence, $(P',p') = (P',p_\sigma) \in \mathcal{D}$. By definition of $\Gamma(P,p)$, all is done if $P' = P_\sigma$ for some $(P,p) \in \mathcal{D}$. Consider the permutation $\sigma^{-1} \in \Sigma$ such that $(p_\sigma)_{\sigma^{-1}} = p$, that is, σ^{-1} is the inverse of σ. By the neutrality of $\mathcal{D}$, $(P',p_\sigma) \in \mathcal{D}$ implies $[P'_{\sigma^{-1}}, (p_\sigma)_{\sigma^{-1}}] = [P'_{\sigma^{-1}}, p] \in \mathcal{D}$. Clearly, $P' = P_\sigma$ for $P = P'_{\sigma^{-1}}$ and $(P,p) \in \mathcal{D}$. Thus $\mathcal{D} \subseteq \cup_{(P,p)\in\mathcal{D}}\Gamma(P,p)$ for an arbitrary $p \in \mathbb{L}$, and the proof is complete. □

We establish below that neutrality for a domain is actually equivalent to the following rank-basedness property.

Definition 3 *A domain $\mathcal{D}$ is rank-based if $\forall (P,p) \in \mathcal{D}$, $\forall p' \in \mathbb{L}$, and $\forall \overrightarrow{a}, \overrightarrow{c}, \overrightarrow{b}, \overrightarrow{d} \in \mathcal{A}$ with $R(\overrightarrow{a}, p) = R(\overrightarrow{c}, p')$ and $R(\overrightarrow{b}, p) = R(\overrightarrow{d}, p')$, there exists $(P',p') \in \mathcal{D}$ such that $\overrightarrow{c}\ P'\ \overrightarrow{d}$ if and only if $\overrightarrow{a}\ P\ \overrightarrow{b}$.*

Lemma 2 *A domain is neutral if and only if it is rank-based.*

Proof Pick a neutral domain $\mathcal{D}$. Take $(P,p) \in \mathcal{D}$, $p' \in \mathbb{L}$, and $\overrightarrow{a}, \overrightarrow{c}, \overrightarrow{b}, \overrightarrow{d} \in \mathcal{A}$ such that $R(\overrightarrow{a}, p) = R(\overrightarrow{c}, p')$ and $R(\overrightarrow{b}, p) = R(\overrightarrow{d}, p')$. For each $m \in \mathcal{M}$, define the permutation σ_m of $\mathcal{A}_m$ by $\forall e_m \in \mathcal{A}_m$, $p'_m(e_m) = p_m[\sigma_m(e_m)]$. By construction, we get $p'_m = p_{m,\sigma_m}$ for all $m \in \mathcal{M}$. Thus $p' = p_\sigma$ where $\sigma = (\sigma_m)_{m\in\mathcal{M}}$. By the neutrality of $\mathcal{D}$, we have $(P_\sigma, p_\sigma) \in \mathcal{D}$. Since $R(\overrightarrow{a}, p) = R(\overrightarrow{c}, p')$ and $R(\overrightarrow{b}, p) = R(\overrightarrow{d}, p')$, then $\sigma(\overrightarrow{a}) = \overrightarrow{c}$ and $\sigma(\overrightarrow{b}) = \overrightarrow{d}$. By definition of P_σ, we get $\overrightarrow{a}\ P\ \overrightarrow{b} \Leftrightarrow \sigma(\overrightarrow{a})\ P_\sigma\ \sigma(\overrightarrow{b})$. Noting that for $P' = P_\sigma$, we have $(P',p') \in \mathcal{D}$ with $\overrightarrow{a}\ P\ \overrightarrow{b} \Leftrightarrow \overrightarrow{c}\ P'\ \overrightarrow{d}$, this establishes that $\mathcal{D}$ is rank-based.

Conversely, let $\mathcal{D}$ be a rank-based domain. Pick any $(P,p) \in \mathcal{D}$. Moreover, take a vector of permutations $\sigma = (\sigma_m)_{m\in\mathcal{M}}$. We want to show that $(P_\sigma, p_\sigma) \in \mathcal{D}$. Take any $\overrightarrow{a}, \overrightarrow{b} \in \mathcal{A}$. By construction, $R(\overrightarrow{a}, p) = R[\sigma(\overrightarrow{a}), p_\sigma]$ and $R(\overrightarrow{b}, p) = R[\sigma(\overrightarrow{b}), p_\sigma]$. Since $\mathcal{D}$ is rank-based, there exists $(P', p_\sigma) \in \mathcal{D}$ such that $\overrightarrow{a}\ P\ \overrightarrow{b} \Leftrightarrow \sigma(\overrightarrow{a})\ P'\ \sigma(\overrightarrow{b})$. It follows that $P' = P_\sigma$, which shows that $(P_\sigma, p_\sigma) \in \mathcal{D}$, and establishes neutrality of $\mathcal{D}$. □

If $\mathcal{D}$ is a neutral domain, we get by lemma 2 that for any $p \in \mathbb{L}$, each $(P, p) \in \mathcal{D}$ generates a unique linear order $\succ_P$ over $\mathcal{R}$, where $\succ_P$ is defined by: $\forall R, R' \in \mathcal{R}$, $R \succ_P R'$ if and only if there exist $\overrightarrow{a}$, $\overrightarrow{b} \in \mathcal{A}$ such that $R(\overrightarrow{a}, p) = R$, $R(\overrightarrow{b}, p) = R'$, and $\overrightarrow{a}\ P\ \overrightarrow{b}$. Combining this with lemma 1, $\mathcal{D}$ is isomorphic to a non-empty subset $\mathcal{D}^*$ of $\mathcal{L}_{\mathcal{R}}$. Pick any coordinate profile $\mathbf{p} = (p^i)_{i \in \mathcal{I}}$ of $(\mathbb{L})^I$. Given a neutral domain $\mathcal{D}$, an outcome profile $\mathbf{P} = (P^i)_{i \in \mathcal{I}} \in (\mathcal{L}_{\mathcal{A}})^I$ is admissible at $\mathbf{p}$ if and only if $\forall \overrightarrow{a}, \overrightarrow{b} \in \mathcal{A}$, $\forall i \in \mathcal{I}$, $\overrightarrow{a}\ P^i\ \overrightarrow{b}$ if and only if there exists $\delta^i \in \mathcal{D}^*$ such that $R(\overrightarrow{a}, p^i)\ \delta^i\ R(\overrightarrow{b}, p^i)$. Hence, each voter i is assigned some admissible linear order δ^i in $\mathcal{L}_{\mathcal{R}}$ which generates an outcome preference from coordinate preference p^i.

This justifies the fact that, with a notational abuse, we henceforth identify $\mathcal{D}$ with $\mathcal{D}^*$. Moreover, we designate elements of $\mathcal{D}$ as preferences that are usually designated by δ.[8] A domain $\mathcal{D}$ plays the role of a reservoir from which each voter can pick a linear order over rank-vectors, which relates in a neutral way coordinate and outcome preferences. If voter i has a coordinate preference p^i and preference δ^i, we write $\delta^i(p^i)$ for the outcome preference P^i such that $(P^i, p^i) \in \mathcal{D}$. Moreover, if $\boldsymbol{\delta} = (\delta^1, \ldots, \delta^I)$ stands for the vector of preferences picked by voters, then we write $\boldsymbol{\delta}(\mathbf{p}) = (\delta^i(p^i))_{i \in \mathcal{I}}$ as the outcome profile associated to the coordinate profile $\mathbf{p}$. Example 1 illustrates how outcomes profiles and coordinate profiles are related given some neutral domain.

Example 1 Pick $M = 2$, $\mathcal{A}_1 = \{a_1, b_1\}$, $\mathcal{A}_2 = \{a_2, b_2\}$, and consider the 3-voter coordinate profile $\mathbf{p}$ having the general form below:

$$\mathbf{p}_1 = \left(\begin{array}{ccc} p_1^1 & p_1^2 & p_1^3 \\ \hline a_1 & b_1 & a_1 \\ b_1 & a_1 & b_1 \end{array} \right), \quad \mathbf{p}_2 = \left(\begin{array}{ccc} p_2^1 & p_2^2 & p_2^3 \\ \hline a_2 & b_2 & b_2 \\ b_2 & a_2 & a_2 \end{array} \right).$$

Take the neutral domain $\mathcal{D} = \{\delta, \delta'\} \subset \mathcal{L}_{\mathcal{R}}$ where

- $\delta = [(1,1)(2,1)(1,2)(2,2)]$, and
- $\delta' = [(1,1)(1,2)(2,1)(2,2)]$.

Suppose that voters 1 and 3 (resp. 2) are assigned to δ (resp. is assigned to δ'). Thus, $\boldsymbol{\delta} = (\delta^1, \delta^2, \delta^3) = (\delta, \delta', \delta)$. The outcome profile associated with $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2)$ looks as below:

$$\boldsymbol{\delta}(\mathbf{p}) = \left(\begin{array}{ccc} \delta(p^1) & \delta^2(p^2) & \delta(p^3) \\ \hline (a_1, a_2) & (b_1, b_2) & (a_1, b_2) \\ (b_1, a_2) & (b_1, a_2) & (b_1, b_2) \\ (a_1, b_2) & (a_1, b_2) & (a_1, a_2) \\ (b_1, b_2) & (a_1, a_2) & (b_1, a_2) \end{array} \right).$$

Based on this notation, we can formally define product stability over a neutral domain.

Definition 4 *A voting rule F is*

(i) (resp. weakly) product stable at the preference $\delta \in \mathcal{L}_{\mathcal{R}}$ if $\forall \mathbf{p} \in \mathbb{L}^I$, F is (weakly) product stable at $((\delta(p^i))_{i \in \mathcal{I}}, \mathbf{p})$,

(ii) (resp. weakly) product stable over the neutral domain $\mathcal{D} \subseteq \mathcal{L}_{\mathcal{R}}$ if $\forall \mathbf{p} \in \mathbb{L}^I$, $\forall \boldsymbol{\delta} \in \mathcal{D}^I$, F is (weakly) product stable at $(\boldsymbol{\delta}(\mathbf{p}), \mathbf{p})$.

Observe that if a voting rule is (weakly) product stable over a neutral domain $\mathcal{D}$, then it is (weakly) product stable at every preference δ belonging to $\mathcal{D}$.

We end this section by defining the property of responsiveness for neutral domains. Given two rank vectors $R, R' \in \mathcal{R}$, we write $R < R'$ if $R \neq R'$ and $r_m \leq r'_m$ for all $m \in \mathcal{M}$.

[8] In Aslan et al. (2021), elements of $\mathcal{L}_{\mathcal{R}}$ are called preference extensions. While the formalization of preference domains differs between their paper and this one, it essentially allows for the same interpretation.

Definition 5 *A preference $\delta \in \mathcal{L}_{\mathcal{R}}$ is responsive if $\forall R, R' \in \mathcal{R}$, $R < R'$ implies $R \; \delta \; R'$. A domain $\mathcal{D}$ is responsive if $\forall \delta \in \mathcal{D}$, δ is responsive.*

We denote by $\mathcal{D}^{resp}$ the neutral and responsive domain. Observe that every two-level preference $(\delta(p), p)$ with $\delta \in \mathcal{D}^{resp}$ is such that $\delta(p)$ is separable with respect to p for any $p \in \mathbb{L}$.[9] More precisely, given a neutral domain $\mathcal{D}$, we have $\forall \delta \in \mathcal{D}$, $\forall p \in \mathbb{L}$, $\delta(p)$ is separable w.r.t. p if and only if δ is responsive.

A typical example of a responsive preference is given by the family of lexicographic preferences. Given a linear order $\widetilde{q} \in \mathcal{L}_{\mathcal{M}}$ and a rank vector $R \in \mathcal{R}$, we write $R^{\widetilde{q}} = (r_1^{\widetilde{q}}, \ldots, r_M^{\widetilde{q}})$ as the rank vector obtained from R by reshuffling its coordinates according to $\widetilde{q}$. Thus, $\forall m \in \mathcal{M}$, we have $r_m^{\widetilde{q}} = r_{\widetilde{q}^{-1}(m)}$. For example, if $M = 3$, $\widetilde{q} = [312]$ and $R = (5, 2, 1)$, then $R^{\widetilde{q}} = (1, 5, 2)$.

Definition 6 *A preference δ is lexicographic w.r.t. $\widetilde{q} \in \mathcal{L}_{\mathcal{M}}$ if $\forall R, R' \in \mathcal{R}$, $R \; \delta \; R'$ if and only if $\exists m^* \in \mathcal{M}$ such that $\forall m \in \{1, \ldots, m^* - 1\}$, $r_m^{\widetilde{q}} = r_m'^{\widetilde{q}}$ and $r_{m^*}^{\widetilde{q}} < r_{m^*}'^{\widetilde{q}}$.*

We denote by $\delta^{\widetilde{q}}$ the lexicographic preference w.r.t. $\widetilde{q} \in \mathcal{L}_{\mathcal{M}}$. Clearly, every lexicographic preference is responsive.

5 Compromise rules

We focus on the specific class of voting rules, $q-$approval rules, with $q \in \{1, 2, \ldots, I\}$. The formal definition of this class of rules requires introducing several notions. Pick any finite set $\mathcal{X}$ with $|\mathcal{X}| = X$ together with a profile $\boldsymbol{\pi} = (\pi^i)_{i \in \mathcal{I}} \in (\mathcal{L}_{\mathcal{X}})^I$. For any $x \in \mathcal{X}$ and any strictly positive integer $r \leq X$, the $r-$support of x at $\boldsymbol{\pi}$ is defined by $S(r, x, \boldsymbol{\pi}) = \left|\{i \in \mathcal{I} : \pi^i(x) \leq r\}\right|$. Hence, $S(r, x, \boldsymbol{\pi})$ is the number of voters for whom x appears among the r best alternatives. Now, for any $q \in \{1, \ldots, I\}$, we define the $q-$value of x at $\boldsymbol{\pi}$ by $V_q(x, \boldsymbol{\pi}) = \min\{r \in \{1, \ldots, X\} : S(r, x, \boldsymbol{\pi}) \geq q\}$. That is, the $q-$value of x at $\boldsymbol{\pi}$ is obtained by pulling down the stick in profile $\boldsymbol{\pi}$ until we reach a rank above which at least q voters place x. Moreover, we define $r(\boldsymbol{\pi}, q) = \min_{x \in \mathcal{X}} V_q(x, \boldsymbol{\pi})$, the minimal rank at $\boldsymbol{\pi}$ for which there exists an alternative that receives the support of at least q voters. Finally, we define $\mathcal{X}(\boldsymbol{\pi}, q) = \arg\min_{x \in \mathcal{X}} V_q(x, \boldsymbol{\pi})$, the set of alternatives at $\boldsymbol{\pi}$ that reach at least q approval at the minimal possible rank. Now we are ready to define the class of $q-$approval (voting) rules formally.

Definition 7 *Given any $q \in \{1, \ldots, I\}$, the $q-$approval rule for $\mathcal{X}$ is the function $F_q : (\mathcal{L}_{\mathcal{X}})^I \rightarrow 2^{\mathcal{X}} \backslash \{\emptyset\}$ defined by: $\forall \boldsymbol{\pi} \in (\mathcal{L}_{\mathcal{X}})^I$, $F_q(\boldsymbol{\pi}) = \{x \in \mathcal{X}(\boldsymbol{\pi}, q) : x \in \arg\max_{y \in \mathcal{X}(\boldsymbol{\pi}, q)} S[r(\boldsymbol{\pi}, q), y, \boldsymbol{\pi}]\}$.*

With words, the $q-$approval rule F_q operates as follows at any profile $\boldsymbol{\pi}$. For each element x of $\mathcal{X}$, the approval cut-off level is lowered one row at a time until a rank above which at least q voters place x is reached. This rank is called the $q-$value of x and is denoted by $V_q(x, \boldsymbol{\pi})$. Then the set $\mathcal{X}(\boldsymbol{\pi}, q)$ formed by all elements with the minimal $q-$value is computed. Finally, we select in $\mathcal{X}(\boldsymbol{\pi}, q)$ all elements with the largest support at this minimal $q-$value. This procedure is illustrated by example 2.

Example 2 Pick $\mathcal{X} = \{a, b, c, d, e\}$, $\mathcal{I} = \{1, \ldots, 10\}$, and define the profile $\boldsymbol{\pi} = (\pi^i)_{i \in \mathcal{I}} \in (\mathcal{L}_{\mathcal{X}})^I$ as below:[10]

[9] An outcome preference $P \in \mathcal{L}_{\mathcal{A}}$ is separable w.r.t. $p = (p_m)_{m \in \mathcal{M}} \in \mathbb{L}$ if and only if $\forall \overrightarrow{a} \in \mathcal{A}$, $\forall m \in \mathcal{M}$, $\forall a_m, b_m \in \mathcal{A}_m$, we have $(a_m, \overrightarrow{a}_{-m}) \; P \; (b_m, \overrightarrow{a}_{-m}) \Leftrightarrow a_m \; p_m \; b_m$.

[10] The first column in $\boldsymbol{\pi}$ should be read as follows: The preference of each of the 4 voters $i = 1, \ldots, 4$ is the linear order $[abcd] \in \mathcal{L}_{\mathcal{X}}$.

$$\boldsymbol{\pi} = \left(\begin{array}{cccc} \pi^i,\, i=1,\dots,4 & \pi^i,\, i=5,6,7 & \pi^i,\, i=8,9 & \pi^{10} \\ \hline a & b & c & d \\ b & c & d & a \\ c & d & a & b \\ d & a & b & c \end{array} \right).$$

For $q \in \{1,\dots,4\}$, we get $r(\boldsymbol{\pi},q)=1$ and $F_q(\boldsymbol{\pi})=\{a\}$. Check that $\mathcal{X}(\boldsymbol{\pi},1)=\mathcal{X}$, $\mathcal{X}(\boldsymbol{\pi},2)=\{a,b,c\}$, $\mathcal{X}(\boldsymbol{\pi},3)=\{a,b\}$, and $\mathcal{X}(\boldsymbol{\pi},4)=\{a\}$. For $q\in\{5,6,7\}$, we get $r(\boldsymbol{\pi},q)=2$ and $F_q(\boldsymbol{\pi})=\{b\}$ (with $\mathcal{X}(\boldsymbol{\pi},5)=\{a,b,c\}$ and $\mathcal{X}(\boldsymbol{\pi},6)=\mathcal{X}(\boldsymbol{\pi},7)=\{b\}$). For $q\in\{8,9\}$, we have $r(\boldsymbol{\pi},q)=3$ and $F_q(\boldsymbol{\pi})=\{c\}$ (with $\mathcal{X}(\boldsymbol{\pi},8)=\{b,c\}$ and $\mathcal{X}(\boldsymbol{\pi},9)=\{c\}$). Finally, $r(\boldsymbol{\pi},10)=4$ and $F_{10}(\boldsymbol{\pi})=\mathcal{X}$.

Two well-known q-compromise rules are the Majoritarian Compromise (Hurwicz and Sertel, 1999; Sertel and Yilmaz, 1999, Núñez and Sanver, 2021) and the Fallback Bargaining (Brams and Kilgour, 2001). The former corresponds to the case where $q=\lceil \frac{I}{2} \rceil$, and the latter to the case where $q=I$.

6 Results

Obviously, (weak) product stability requires the existence of a logical link between the two levels of preferences. As a consequence, one cannot expect a compromise to be weakly product stable at all two-level profiles. As an illustration, we provide a simple example showing that, in two-dimension combinatorial voting with three voters and two alternatives per coordinate, the majoritarian compromise $F_{\lceil \frac{I}{2} \rceil}$ is not weakly product stable over the responsive and neutral domain $\mathcal{D}^{resp}$.

Example 3 Consider again example 1. It is obvious to see that $F_{\lceil \frac{I}{2} \rceil}(\mathbf{p}_1)\times F_{\lceil \frac{I}{2} \rceil}(\mathbf{p}_2)] = \{(a_1,b_2)\}$. Moreover, note that δ and δ' are responsive. Finally, the definition of $\boldsymbol{\delta}(\mathbf{p})$ ensures that $F_{\lceil \frac{I}{2} \rceil}[\boldsymbol{\delta}(\mathbf{p})] = \{(b_1,a_2),(b_1,b_2)\}$. Since $[F_{\lceil \frac{I}{2} \rceil}(\mathbf{p}_1)\times F_{\lceil \frac{I}{2} \rceil}(\mathbf{p}_2)] \cap F_{\lceil \frac{I}{2} \rceil}[\boldsymbol{\delta}(\mathbf{p})]=\emptyset$, $F_{\lceil \frac{I}{2} \rceil}$ is not weakly product stable over $\mathcal{D}=\{\delta,\delta'\}$. Since $\mathcal{D}\subset\mathcal{D}^{resp}$, then $F_{\lceil \frac{I}{2} \rceil}$ is not weakly product stable over $\mathcal{D}^{resp}$.

We investigate the existence of neutral domains over which $q-$approval rules are (weakly) product stable, where q can take any value between 1 and I.

We denote by $\overrightarrow{1}$ the element $(\overbrace{1,\dots,1}^{M})$ of $\mathcal{R}$. Our first result establishes that regardless of the value of q, F_q is weakly product stable over a neutral domain $\mathcal{D}$ only if $\mathcal{D}$ satisfies a simple condition: every preference in $\mathcal{D}$ must rank $\overrightarrow{1}$ first.

Lemma 3 *Take any $I\geq 2$ and any $q\in\{1,\dots,I\}$. Then F_q is weakly product stable over a neutral domain $\mathcal{D}$ only if every preference $\delta\in\mathcal{D}$ ranks $\overrightarrow{1}$ first.*

Proof Suppose that there exists a neutral preference $\delta\in\mathcal{D}$ such that R is ranked first for some $R\in\mathcal{R}\backslash\{\overrightarrow{1}\}$. Pick any $I>1$ and consider the coordinate profile $\mathbf{p}=(p^i)_{i\in\mathcal{I}}$ such that $\forall m\in\mathcal{M}$, $\forall i\in\mathcal{I}$,
- $p^i_m(a_m)=1$,
- if $r_m\neq 1$, $\exists b_m\in\mathcal{A}_m\setminus\{a_m\}$ such that $p^i_m(b_m)=r_m$.

Obviously, $F_q(\mathbf{p}_m)=a_m$ for all $m\in\mathcal{M}$ and all $q\in\{1,\dots,I\}$. Thus, $\prod_{m\in\mathcal{M}}F_q(\mathbf{p}_m)=\{\overrightarrow{a}\}$ where $\overrightarrow{a}=(a_1,\dots,a_M)$. Define $\overrightarrow{c}=(c_1,\dots,c_M)\in\mathcal{A}$ by $\forall m\in\mathcal{M}$, $c_m=\left\{\begin{array}{ll} a_m & \text{if } r_m=1 \\ b_m & \text{if } r_m\neq 1\end{array}\right.$.

Now, take the vector of extension rules $\boldsymbol{\delta}=(\delta^1,\dots,\delta^I)=(\delta,\delta,\dots,\delta)$. Since $R(\overrightarrow{c},p^i)=R$ for all $i\in\mathcal{I}$ and R is top-ranked by δ, then $F_q\left[\boldsymbol{\delta}(\mathbf{p})\right]=\{\overrightarrow{c}\}$. Therefore, $[\prod_{m\in\mathcal{M}}F_q(\mathbf{p}_m)]\cap F_q\left[\boldsymbol{\delta}(\mathbf{p})\right]=\emptyset$, which shows that F_q is not weakly product stable over $\mathcal{D}$.

In the rest of this section, we successively analyze product stability for the Fallback Bargaining rule (section 4.1) and $q-$approval rules with $q \neq I$ (section 4.2).

6.1 Fallback bargaining

Theorem 1 below characterizes neutral domains over which the Fallback Bargaining rule F_I is weakly stable, provided a large enough number of voters. In order to establish this theorem, we proceed in several steps. The first two steps are useful observations.

Lemma 4 *Let $\hat{I} > \overline{I} \geq 2$. If there exists a profile with $\overline{I}$ voters at which F_I fails (resp. weak) product stability at preference δ, then there exists a profile with $\hat{I}$ voters at which F_I fails (resp. weak) product stability at δ.*

Proof Take a coordinate profile $\mathbf{p} \in \mathbb{L}^{\overline{I}}$ at which F_I fails weak product stability at δ. Pick any $\hat{I} > \overline{I}$ and define $\mathbf{p}' \in \mathbb{L}^{\hat{I}}$ by

- $\forall i \in \{1, \ldots, \overline{I}\}$, $p'^i = p^i$,
- $\forall i \in \{\overline{I}+1, \ldots, \hat{I}\}$, $p'^i = p^1$.

It is straightforward to check that $\prod_{m \in \mathcal{M}} F_I(\mathbf{p}_m) = \prod_{m \in \mathcal{M}} F_I(\mathbf{p}'_m)$ while $F_I\left[\boldsymbol{\delta}(\mathbf{p})\right] = F_I\left[\boldsymbol{\delta}'(\mathbf{p}')\right]$, where $\boldsymbol{\delta} = (\overbrace{\delta, \ldots, \delta}^{\overline{I}})$ and $\boldsymbol{\delta}' = (\overbrace{\delta, \ldots, \delta}^{\hat{I}})$. If $[\prod_{m \in \mathcal{M}} F_I(\mathbf{p}_m)] \cap F_I\left[\boldsymbol{\delta}(\mathbf{p})\right] = \emptyset$, then F_I fails weak product stability at δ and $\mathbf{p}'$. Similarly, if $[\prod_{m \in \mathcal{M}} F_I(\mathbf{p}_m)] \nsubseteq F_I\left[\boldsymbol{\delta}(\mathbf{p})\right]$, then F_I fails product stability at δ and $\mathbf{p}'$, which completes the proof.

Lemma 5 *Let $\delta \in \mathcal{L}_{\mathcal{R}}$ be a preference that fails responsiveness. Then there exists $m^* \in \mathcal{M}$, $\bar{R} = (r_1, \ldots, r_M) \in \mathcal{R}$ and $\bar{R}' = (r'_{m^*}, R_{-m^*}) \in \mathcal{R}$ such that $r_{m^*} < r'_{m^*}$, and $\bar{R}'\ \delta\ \bar{R}$.*

Proof If δ fails responsiveness, then there exist two rank vectors $R = (r_1, \ldots, r_M) \in \mathcal{R}$ and $R' = (r'_1, \ldots, r'_M) \in \mathcal{R}$ such that $R < R'$, and $R'\ \delta\ R$. Define $\mathcal{M}' = \{m \in \mathcal{M} : r_m < r'_m\}$. If $|\mathcal{M}'| = 1$, let $\bar{R} = R$ and $\bar{R}' = R'$, and we are done. If $|\mathcal{M}'| > 1$ pick any $m^1 \in \mathcal{M}'$ and consider $\mathcal{M}^1 = \mathcal{M}' \setminus \{m^1\}$. Let $R^1 = (r_{m^1}, R'_{-m^1})$, either $[R^1\ \delta\ R$ and $\{m \in \mathcal{M} : R^1_m \neq R\} = \mathcal{M}^1]$ or $[R\ \delta\ R^1$, by transitivity $R'\ \delta\ R^1$ and $\{m \in \mathcal{M} : R^1_m \neq R'\} = \{m^1\}]$. In both cases, we constructed two new vectors at which δ violates responsiveness, and the number of coordinates for which they differ is at most $|\mathcal{M}'| - 1$. As there are finitely many coordinates, repeating the same argument at most $M - 1$ times gives the desired result.

The third step consists of showing that if the number of voters is large enough, F_I is weakly product stable at a preference δ only if δ is responsive. Pick any coordinate $m^* \in \mathcal{M}$ such that $m^* \in \arg\min_{m \in \mathcal{M}} A_m$, and define $I^* = \prod_{m \in \mathcal{M} \setminus \{m^*\}} A_m$.

Proposition 1 *If $I \geq I^*$, F_I is weakly product stable over $\mathcal{D}$ only if $\mathcal{D} \subseteq \mathcal{D}^{resp}$.*

Proof Suppose that F_I is weakly product stable at $\delta \notin \mathcal{D}^{resp}$. By lemma 5, there exists $m^* \in \mathcal{M}$, $R = (r_1, \ldots, r_M) \in \mathcal{R}$ and $R' = (r'_{m^*}, R_{-m^*}) \in \mathcal{R}$ such that $r_{m^*} < r'_{m^*}$, and $R'\ \delta\ R$. Without loss of generality, we can assume $m^* = 1$. Define $\mathcal{M}^1 = \{m \in \mathcal{M} : 1 = r_m = r'_m\}$, and $\mathcal{M}^2 = \{m \in \mathcal{M} : 1 < r_m = r'_m\}$. Thus, $R = (r_1, \mathbf{1}_{\mathcal{M}^1}, R_{\mathcal{M}^2})$ and $R' = (r'_1, \mathbf{1}_{\mathcal{M}^1}, R_{\mathcal{M}^2})$ such that $r_1 < r'_1$, and $R'\ \delta\ R$. We consider two cases. In both cases, we exhibit a coordinate profile with I^* voters and show that F_I fails weak product stability at δ.

Case 1: $r_1 > 1$

Pick a I^*-voter coordinate profile $\mathbf{p}$ having the general form below[11]:

- $\mathbf{p}_1 = \left(\begin{array}{ccc} rank & p_1^1 & p_1^i,\, i \geq 2 \\ \hline 1 & \cdot & a_1 \\ \dots & \dots & \dots \\ r_1 & b_1 & b_1 \\ \dots & \dots & \dots \\ r_1' & a_1 & \cdot \\ \dots & \dots & \dots \end{array}\right)$,

- $\forall m \in \mathcal{M}^1$, $\forall i \in \mathcal{I}$, $p_m^i(a_m) = 1 = r_m = r_m'$,

- $\forall m \in \mathcal{M}^2$, $\mathbf{p}_m = \left(\begin{array}{ccc} rank & p_m^1 & p_m^i,\, i \geq 2 \\ \hline 1 & \cdot & a_m \\ \dots & \dots & \dots \\ r_m = r_m' & a_m & \cdot \\ \dots & \dots & \dots \end{array}\right)$.

Now, define $\mathcal{A}' = \prod_{m \in \mathcal{M}} \mathcal{A}_m'$ where $\mathcal{A}_m' = \begin{cases} \{b_1\} & \text{if } m = 1 \\ \{a_m\} & \text{if } m \in \mathcal{M}^1 \\ \mathcal{A}_m & \text{if } m \in \mathcal{M}^2 \end{cases}$. Observe that $|\mathcal{A}'| = \prod_{m \in \mathcal{M}^2} A_m \leq I^*$ the last inequality follows from $\mathcal{M}^2 \subseteq \mathcal{M} \backslash \{1\}$. Now, we further specify $\mathbf{p}$ so that it satisfies the following additional properties:

- for $m = 1$, $\forall d_1 \in \{c_1 \in \mathcal{A}_1 : p_1^1(c_1) < r_1\}$, there exits at least one voter $i \in \mathcal{I} \setminus \{1\}$ such that $p^i(d_1) = r_1'$,

- $\forall \overrightarrow{c} \in \mathcal{A}'$, there exists at least one voter $i \in \mathcal{I}$ such that $R(\overrightarrow{c}, p^i) = R$.

It is easy to check that $F_I(\mathbf{p}_1) = \{b_1\}$, $F_I(\mathbf{p}_m) = \{a_m\}$ for all $m \in \mathcal{M}^1$, and $F_I(\mathbf{p}_m) \subseteq \mathcal{A}_m$ for all $m \in \mathcal{M}^2$, that is $[\prod_{m \in \mathcal{M}} F_I(\mathbf{p}_m)] \subseteq \mathcal{A}'$. Moreover, by construction of $\mathbf{p}$, for any $\overrightarrow{c} \in \prod_{m \in \mathcal{M}} F_I(\mathbf{p}_m)$, there exists a voter $i \in \mathcal{I}$ such that $R(\overrightarrow{c}, p^i) = R$. Let $\overrightarrow{a} = (a_1, \dots, a_M) \in \mathcal{A}$, note that $\overrightarrow{a} \notin \mathcal{A}'$, and by construction of $\mathbf{p}$, $R(\overrightarrow{a}, p^i) \in \{R', \overrightarrow{1}\}$ for all $i \in \mathcal{I}$. Since F_I is weakly product stable at δ, then δ must rank $\overrightarrow{1}$ first by lemma 3. Moreover, we must have $R'\ \delta\ R$. Therefore, $V_I[\overrightarrow{a}, \boldsymbol{\delta}(\mathbf{p})] < V_I[\overrightarrow{c}, \boldsymbol{\delta}(\mathbf{p})]$ for any $\overrightarrow{c} \in \prod_{m \in \mathcal{M}} F_I(\mathbf{p}_m)$, and $[\prod_{m \in \mathcal{M}} F_I(\mathbf{p}_m)] \cap F_I[\boldsymbol{\delta}(\mathbf{p})] = \emptyset$, which contradicts with F_I being weakly product stable at δ.

Case 2: $r_1 = 1$

First, note that by lemma 3 and $R'\ \delta\ R$, we have $R \neq \mathbf{1}$. Combined with $r_1 = 1$, this implies that $\mathcal{M}^2 \neq \emptyset$. By construction, we have $R = (r_1, \mathbf{1}_{\mathcal{M}^1}, R_{\mathcal{M}^2})$. Define $\bar{R}' = (\mathbf{1}_{\mathcal{M}^2}, R'_{-\mathcal{M}^2}) = (r_1', \mathbf{1}_{\mathcal{M}^1}, \mathbf{1}_{\mathcal{M}^2})$, and consider the following two subcases:

Case 2a: $\bar{R}'\ \delta\ R$

Pick a I^*-voter coordinate profile $\mathbf{p}$ having the general form below:

- $\mathbf{p}_1 = \left(\begin{array}{ccc} rank & p_1^1 & p_1^i,\, i \geq 2 \\ r_1 = 1 & b_1 & b_1 \\ \dots & \dots & \dots \\ r_1' & a_1 & a_1 \\ \dots & \dots & \dots \end{array}\right)$,

[11] Profiles in matrix form are completed by a column indicating the position in each voter's ranking of elements that matter in the reasoning. Cells that are left blank can be arbitrarily filled by one element among the remaining ones.

- $\forall m \in \mathcal{M}^1$, $\forall i \in \mathcal{I}$, $p^i_m(a_m) = 1 = r_m = r'_m$,

- $\forall m \in \mathcal{M}^2$, $\mathbf{p}_m = \begin{pmatrix} rank & p^1_m & p^i_m,\ i \geq 2 \\ \hline 1 & \cdot & a_m \\ \ldots & \ldots & \ldots \\ r_m = r'_m & a_m & \cdot \\ \ldots & \ldots & \ldots \end{pmatrix}$.

Now, define $\mathcal{A}' = \prod_{m\in\mathcal{M}} \mathcal{A}'_m$ where $\mathcal{A}'_m = \begin{cases} \{b_1\} & \text{if } m = 1 \\ \{a_m\} & \text{if } m \in \mathcal{M}^1 \\ \mathcal{A}_m & \text{if } m \in \mathcal{M}^2 \end{cases}$. Observe that $|\mathcal{A}'| = \prod_{m\in\mathcal{M}^2} A_m \leq I^*$. As for case 1, we further specify $\mathbf{p}$ so that it satisfies the following additional property:

- $\forall \overrightarrow{c} \in \mathcal{A}'$, there exists at least one voter $i \in \mathcal{I}$ such that $R(\overrightarrow{c}, p^i) = R$.

It is easy to check that $F_I(\mathbf{p}_1) = \{b_1\}$, $F_I(\mathbf{p}_m) = \{a_m\}$ for all $m \in \mathcal{M}^1$, and $F_I(\mathbf{p}_m) \subseteq \mathcal{A}_m$ for all $m \in \mathcal{M}^2$, that is $[\prod_{m\in\mathcal{M}} F_I(\mathbf{p}_m)] \subseteq \mathcal{A}'$. Moreover, by construction of $\mathbf{p}$, for any $\overrightarrow{c} \in \prod_{m\in\mathcal{M}} F_I(\mathbf{p}_m)$, there exists a voter $i \in \mathcal{I}$ such that $R(\overrightarrow{c}, \overline{p}^i) = R$. Let $\overrightarrow{a} = (a_1, \ldots, a_M) \in \mathcal{A}$, note that $\overrightarrow{a} \notin \mathcal{A}'$, and by construction of $\mathbf{p}$, $R(\overrightarrow{a}, p^i) \in \{R', \bar{R}'\}$ for all $i \in \mathcal{I}$. By definition $R'\ \delta\ R$ and we are in the case of $\bar{R}'\ \delta\ R$. Thus $V_I[\overrightarrow{a}, \boldsymbol{\delta}(\mathbf{p})] < V_I[\overrightarrow{c}, \boldsymbol{\delta}(\mathbf{p})]$ for any $\overrightarrow{c} \in \prod_{m\in\mathcal{M}} F_I(\mathbf{p}_m)$, and $[\prod_{m\in\mathcal{M}} F_I(\mathbf{p}_m)] \cap F_I[\boldsymbol{\delta}(\mathbf{p})] = \emptyset$, which contradicts with F_I being weakly product stable at δ.

Case 2b: $R\ \delta\ \bar{R}'$

Pick a I^*-voter coordinate profile $\mathbf{p}$ having the general form below:

- $\mathbf{p}_1 = \begin{pmatrix} rank & p^1_1 & p^i_1,\ i \geq 2 \\ r_1 = 1 & \cdot & a_1 \\ \ldots & \ldots & \ldots \\ r'_1 & a_1 & \cdot \\ \ldots & \ldots & \ldots \end{pmatrix}$,

- $\forall m \in \mathcal{M}^1$, $\forall i \in \mathcal{I}$, $p^i_m(a_m) = 1 = r_m = r'_m$,

- $\forall m \in \mathcal{M}^2$, $\mathbf{p}_m = \begin{pmatrix} rank & p^1_m & p^i_m,\ i \geq 2 \\ \hline 1 & b_m & b_m \\ \ldots & \ldots & \ldots \\ r_m = r'_m & a_m & a_m \\ \ldots & \ldots & \ldots \end{pmatrix}$.

Now, define $\mathcal{A}' = \prod_{m\in\mathcal{M}} \mathcal{A}'_m$ where $\mathcal{A}'_m = \begin{cases} \mathcal{A}_1 & \text{if } m = 1 \\ \{a_m\} & \text{if } m \in \mathcal{M}^1 \\ \{b_m\} & \text{if } m \in \mathcal{M}^2 \end{cases}$. Observe that $|\mathcal{A}'| = A_1 \leq I^*$. As for case 2a, we further specify $\mathbf{p}$ so that it satisfies the following additional property:

- $\forall \overrightarrow{c} \in \mathcal{A}'$, there exists at least one voter $i \in \mathcal{I}$ such that $R(\overrightarrow{c}, p^i) = \bar{R}'$.

It is easy to check that $F_I(\mathbf{p}_1) = \{b_1\}$, $F_I(\mathbf{p}_m) = \{a_m\}$ for all $m \in \mathcal{M}^1$, and $F_I(\mathbf{p}_m) \subseteq \mathcal{A}_m$ for all $m \in \mathcal{M}^2$, that is $[\prod_{m\in\mathcal{M}} F_I(\mathbf{p}_m)] \subseteq \mathcal{A}'$. Moreover, by construction of $\mathbf{p}$, for any $\overrightarrow{c} \in \prod_{m\in\mathcal{M}} F_I(\mathbf{p}_m)$, there exists a voter $i \in \mathcal{I}$ such that $R(\overrightarrow{c}, p^i) = \bar{R}'$. Let $\overrightarrow{a} = (a_1, \ldots, a_M) \in \mathcal{A}$, note that $\overrightarrow{a} \notin \mathcal{A}'$, and by construction of $\mathbf{p}$, $R(\overrightarrow{a}, p^i) \in \{R', R\}$ for all $i \in \mathcal{I}$. By definition $R'\ \delta\ R$ and we are in the case of $R\ \delta\ \bar{R}'$, by transitivity, we get $R', R\ \delta\ \bar{R}'$. Thus $V_I[\overrightarrow{a}, \boldsymbol{\delta}(\mathbf{p})] < V_I[\overrightarrow{c}, \boldsymbol{\delta}(\mathbf{p})]$ for any $\overrightarrow{c} \in \prod_{m\in\mathcal{M}} F_I(\mathbf{p}_m)$, and $[\prod_{m\in\mathcal{M}} F_I(\mathbf{p}_m)] \cap F_I[\boldsymbol{\delta}(\mathbf{p})] = \emptyset$, which contradicts with F_I being weak product stable at δ.

This shows that F_I fails weak product stability at δ in restriction to profiles with I^* voters. We complete the proof by using lemma 4.

It is worth observing that F_I may fail weak product stability at some responsive preference, as shown by the following example.

Example 4 Let $M = I = 2$ and take a coordinate profile $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2)$ such as below:

$$\mathbf{p}_1 = \left(\begin{array}{ccc} rank & p_1^1 & p_1^2 \\ \hline 1 & a_1 & c_1 \\ 2 & b_1 & b_1 \\ 3 & c_1 & a_1 \\ \dots & \dots & \dots \end{array} \right), \mathbf{p}_2 = \left(\begin{array}{ccc} rank & p_2^1 & p_2^2 \\ \hline 1 & a_2 & c_2 \\ 2 & b_2 & b_2 \\ 3 & c_2 & a_2 \\ \dots & \dots & \dots \end{array} \right).$$

Take preference $\delta = [(1,1)(1,2)(2,1)(1,3)(3,1)(2,2)(2,3)(3,2)(3,3)...] \in \mathcal{L}_\mathcal{R}$. It is straightforward to check that $\delta \in \mathcal{D}^{resp}$. Clearly, $[F_I(\mathbf{p}_1) \times F_I(\mathbf{p}_2)] = \{(b_1, b_2)\}$. The outcome profile associated with $\boldsymbol{\delta} = (\delta, \delta)$ is

$$\boldsymbol{\delta}(\mathbf{p}) = \left(\begin{array}{ccc} rank & \delta(\mathbf{p}^1) & \delta(\mathbf{p}^2) \\ \hline 1 & (a_1, a_2) & (c_1, c_2) \\ 2 & (a_1, b_2) & (c_1, b_2) \\ 3 & (b_1, a_2) & (b_1, c_2) \\ 4 & (a_1, c_2) & (c_1, a_2) \\ 5 & (c_1, a_2) & (a_1, c_2) \\ 6 & (b_1, b_2) & (b_1, b_2) \\ \dots & \dots & \dots \end{array} \right).$$

Since $F_I[\boldsymbol{\delta}(\mathbf{p})] = \{(a_1, c_2), (c_1, a_2)\}$, F_I is not weakly product stable at δ.

Actually, F_I is weakly product stable only over small subsets of $\mathcal{D}^{resp}$, each comprising as unique element a preference that is lexicographic w.r.t. some priority order over coordinates.

Theorem 1 *If $I \geq I^* + 3$, F_I is weakly product stable over a neutral domain $\mathcal{D}$ if and only if $\mathcal{D} = \{\delta^{\widetilde{q}}\}$ where $\delta^{\widetilde{q}}$ is lexicographic w.r.t. $\widetilde{q} \in \mathcal{L}_\mathcal{M}$.*

The proof of theorem 1 is provided in the Appendix. Note that the sufficiency part of theorem 1 holds for any number of voters. Moreover, as stated in proposition 2 below, I^* is actually the lower bound for responsiveness to be necessary.

Proposition 2 *If $I < I^*$, then there exists a preference $\delta \in \mathcal{L}_\mathcal{R} \setminus \mathcal{D}^{resp}$ for which F_I is weakly product stable.*

Proof Without loss of generality assume $M \in \arg\min_{m \in \mathcal{M}} A_m$, thus $I^* = \prod_{m \in \mathcal{M} \setminus \{M\}} A_m$. Take $I < I^*$. Let $R = (A_1, A_2, \dots, A_M), R' = (A_1, A_2, \dots, A_{M-1}, A_M - 1) \in \mathcal{R}$, and define preference δ: If $\mathcal{R}^* = \mathcal{R} \setminus \{R, R'\}$,

- $\delta|_{\mathcal{R}^* \times \mathcal{R}^*} = \delta^{\widetilde{q}}|_{\mathcal{R}^* \times \mathcal{R}^*}$ where $\widetilde{q} = [12 \dots M]$,
- $\forall R'' \in \mathcal{R}^*$, $R'' \; \delta \; R \; \delta \; R'$.

Hence, δ is obtained from the lexicographic preference w.r.t natural order over coordinates by swapping the two bottom rank vectors $R = (A_1, A_2, \dots, A_M)$ and $R' = (A_1, A_2, \dots, A_{M-1}, A_M - 1)$. Clearly, $\delta \notin \mathcal{D}^{resp}$.

Consider any coordinate profile $\mathbf{p}$ with $2 \leq I < I^*$ voters and $\boldsymbol{\delta} = (\delta, \delta, \dots, \delta)$. Suppose, towards a contradiction, that $[\prod_{m \in \mathcal{M}} F_I(\mathbf{p}_m)] \cap F_I[\boldsymbol{\delta}(\mathbf{p})] = \emptyset$. We establish the result by showing three claims.

Claim 1: For all $\overrightarrow{a} \in \prod_{m \in \mathcal{M}} F_I(\mathbf{p}_m)$, for all $i \in \mathcal{I}$, $R(\overrightarrow{a}, p^i) \neq R = (A_1, A_2, \dots, A_M)$.

Proof of Claim 1: If there exists $\overrightarrow{a} \in \prod_{m \in \mathcal{M}} F_I(\mathbf{p}_m)$ and there exists a voter $i_{\overrightarrow{a}} \in \mathcal{I}$ such that $R(\overrightarrow{a}, p^{i_{\overrightarrow{a}}}) = R$, then $\prod_{m \in \mathcal{M}} F_I(\mathbf{p}_m) = \mathcal{A}$, leading to $\mathcal{A} \cap F_I[\boldsymbol{\delta}(\mathbf{p})] = \emptyset$ which is not possible.

Claim 2: Consider the coordinate profile $\mathbf{p}$ with extension profile $\boldsymbol{\delta}^{\widetilde{q}}=(\delta^{\widetilde{q}},\delta^{\widetilde{q}},\ldots,\delta^{\widetilde{q}})$. There exists $\overrightarrow{a} \in [\prod_{m\in\mathcal{M}} F_I(\mathbf{p}_m)] \cap F_I\left[\boldsymbol{\delta}^{\widetilde{q}}(\mathbf{p})\right]$ such that there exists a voter $i_{\overrightarrow{a}} \in \mathcal{I}$ with $R(\overrightarrow{a}, p^{i_{\overrightarrow{a}}}) = R'$.
Note that this implies $r(\boldsymbol{\delta}^{\widetilde{q}}(\mathbf{p}), I) = \min_{\overrightarrow{a}\in\mathcal{A}} V_I[\overrightarrow{a}, \boldsymbol{\delta}^{\widetilde{q}}(\mathbf{p})] = \left(\prod_{m\in\mathcal{M}} A_m\right) - 1$.

Proof of Claim 2: By sufficiency part of theorem 1, which holds for any I value, $[\prod_{m\in\mathcal{M}} F_I(\mathbf{p}_m)] \cap F_I\left[\boldsymbol{\delta}^{\widetilde{q}}(\mathbf{p})\right] \neq \emptyset$. Pick $\overrightarrow{b} \in [\prod_{m\in\mathcal{M}} F_I(\mathbf{p}_m)] \cap F_I\left[\boldsymbol{\delta}^{\widetilde{q}}(\mathbf{p})\right]$, by Claim 1, $\overrightarrow{b} \in \prod_{m\in\mathcal{M}} F_I(\mathbf{p}_m)$ implies $\forall i \in \mathcal{I}$, $R(\overrightarrow{b}, p^i) \neq R$. Suppose towards a contradiction that $\forall i \in \mathcal{I}$, $R(\overrightarrow{b}, p^i) \in \mathcal{R}^* = \mathcal{R} \setminus \{R, R'\}$. Noting that $\overrightarrow{b} \in F_I\left[\boldsymbol{\delta}^{\widetilde{q}}(\mathbf{p})\right]$, by definition of δ, $[\forall i \in \mathcal{I}, R(\overrightarrow{b}, p^i) \in \mathcal{R}^*]$ implies $\overrightarrow{b} \in F_I\left[\boldsymbol{\delta}(\mathbf{p})\right]$. This clearly contradicts with $[\prod_{m\in\mathcal{M}} F_I(\mathbf{p}_m)] \cap F_I\left[\boldsymbol{\delta}(\mathbf{p})\right] = \emptyset$.

Claim 3: For all $\overrightarrow{a} \in \prod_{m\in\mathcal{M}} F_I(\mathbf{p}_m)$ and there exists a voter $i_{\overrightarrow{a}} \in \mathcal{I}$ such that $R(\overrightarrow{a}, p^{i_{\overrightarrow{a}}}) = R'$.

Proof of Claim 3: Suppose that there exists $\overrightarrow{b} \in \prod_{m\in\mathcal{M}} F_I(\mathbf{p}_m)$ such that $\forall i \in \mathcal{I}$, $R(\overrightarrow{b}, p^i) \neq R'$. By Claim 1, $\forall i \in \mathcal{I}$, $R(\overrightarrow{b}, p^i) \neq R$. Since $[\forall i \in \mathcal{I}, R(\overrightarrow{b}, p^i) \neq R']$, then $\forall i \in \mathcal{I}$, $R(\overrightarrow{b}, p^i) \in \mathcal{R}^* = \mathcal{R} \setminus \{R, R'\}$. Noting that by definition of $\delta^{\widetilde{q}}$ for all $R'' \in \mathcal{R}^*$, $R''\ \delta^{\widetilde{q}}\ R\ \delta^{\widetilde{q}}\ R'$, we get $V_I[\overrightarrow{b}, \boldsymbol{\delta}^{\widetilde{q}}(\mathbf{p})] < \left(\prod_{m\in\mathcal{M}} A_m\right) - 1$. By Claim 2, $r(\boldsymbol{\delta}^{\widetilde{q}}(\mathbf{p}), I) = \left(\prod_{m\in\mathcal{M}} A_m\right) - 1$, which brings the desired contradiction.

Finally, for all $m \in \mathcal{M}\backslash\{M\}$, we have $F_I(\mathbf{p}_m) = \mathcal{A}_m$. It follows that $\left|\prod_{m\in\mathcal{M}} F_I(\mathbf{p}_m)\right| \geq \prod_{m\in\mathcal{M}\backslash\{M\}} A_m = I^*$. Combining claim 3 combined with $I < I^*$ gives the desired contradiction.

Whether theorem 1 can be similarly stated for less than $I^* + 3$ voters is left as an open question.

Not surprisingly, strengthening weak product stability to product stability leads to a negative result.

Theorem 2 *If $I \geq I^* + 3$, F_I is product stable at no preference δ.*

Proof Pick any preference δ at which F_I is product stable. Since product stability implies weak product stability, theorem 1 implies that $\delta = \delta^{\widetilde{q}}$ for some $\widetilde{q} \in \mathcal{L}_{\mathcal{M}}$. Hence, it is sufficient to show that for any $\widetilde{q} \in \mathcal{L}_{\mathcal{M}}$, F_I is not product stable at $\delta^{\widetilde{q}}$. Up to a reshuffling of coordinates, attention can be restricted to $\widetilde{q} = [M \ldots 21]$. Take any $I \geq 2$, any $M \geq 2$, and any $A_m \geq 2$ for all $m \in \mathcal{M}$. Define the coordinate profile $\mathbf{p}$ as below:

$$- \mathbf{p}_1 = \left(\begin{array}{ccc} rank & p_1^1 & p_1^i,\ i \geq 2 \\ \hline 1 & b_1 & a_1 \\ 2 & a_1 & b_1 \\ \ldots & \ldots & \ldots \end{array}\right),\ \mathbf{p}_2 = \left(\begin{array}{ccc} rank & p_2^1 & p_2^i,\ i \geq 2 \\ \hline 1 & b_2 & a_2 \\ 2 & a_2 & b_2 \\ \ldots & \ldots & \ldots \end{array}\right)$$

$$- \forall m > 2,\ \mathbf{p}_m = \left(\begin{array}{cc} rank & p_m^i,\ i \geq 1 \\ \hline 1 & a_m \\ \ldots & \ldots \end{array}\right).$$

Obviously, $\prod_{m\in\mathcal{M}} F_I(\mathbf{p}_m) = \{a_1, b_1\} \times \{a_2, b_2\} \times \{\overrightarrow{a}_{>2}\}$, where $\overrightarrow{a}_{>2} = (a_3, \ldots, a_M)$. Take $\boldsymbol{\delta} = (\delta^{\widetilde{q}}, \ldots, \delta^{\widetilde{q}})$. By definition of $\delta^{\widetilde{q}}$, the outcome profile is

$$\boldsymbol{\delta}(\mathbf{p}) = \left(\begin{array}{c|cc} rank & \delta^{\widetilde{q}}(p^1) & \delta(p^i),\ i \geq 2 \\ \hline 1 & (b_1, b_2, \overrightarrow{a}_{>2}) & (a_1, a_2, \overrightarrow{a}_{>2}) \\ 2 & (a_1, b_2, \overrightarrow{a}_{>2}) & (b_1, a_2, \overrightarrow{a}_{>2}) \\ \ldots & \mathcal{C}_1 & \mathcal{C}_2 \\ A_1 + 1 & (b_1, a_2, \overrightarrow{a}_{>2}) & (a_1, b_2, \overrightarrow{a}_{>2}) \\ \ldots & \ldots & \ldots \end{array}\right),$$

where $\mathcal{C}_1 = \{(c, b_2, \overrightarrow{a}_{>2}) \in \mathcal{A} : c \in \mathcal{A}_1 \setminus \{a_1, b_1\}\}$ and $\mathcal{C}_2 = \{(c, a_2, \overrightarrow{a}_{>2}) \in \mathcal{A} : c \in \mathcal{A}_1 \setminus \{a_1, b_1\}\}$. Since $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$, then $F_I[\boldsymbol{\delta}(\mathbf{p})] = \{(b_1, a_2, \overrightarrow{a}_{>2}), (a_1, b_2, \overrightarrow{a}_{>2})\}$. It follows that $[\prod_{m\in\mathcal{M}} F_I(\mathbf{p}_m)] \nsubseteq F_I[\boldsymbol{\delta}(\mathbf{p})]$. Therefore F_I is not product stable at $\delta^{\widetilde{q}}$. The conclusion follows from lemma 4.

6.2 $q-$approval rules with $q \neq I$

We turn to $q-$approval rules with $q < I$. It turns out that even weak product stability cannot be satisfied unless very special cases are considered. First, we consider rules with $\frac{I}{2} < q < I$. Observe that this condition implies $I \geq 3$.

Proposition 3 *Take any $I \geq 3$. If $\frac{I}{2} < q < I$, F_q is weakly product stable at no preference δ.*

Proof Let δ be a preference at which F_q is weakly product stable, with $\frac{I}{2} < q < I$. By lemma 3, δ must rank $\overrightarrow{1}$ first. Denote by $\widetilde{R}^\delta = (r_1^\delta, \ldots, r_M^\delta)$ the second-best rank vector for δ. Since $\widetilde{R}^\delta \neq \overrightarrow{1}$, we can assume w.l.o.g. that $r_1^\delta > 1$.

Case 1: $\widetilde{R}^\delta = (r_1^\delta, \overbrace{1, \ldots, 1}^{M-1})$.

Consider a coordinate profile $\mathbf{p}$ with $I \geq 3$ voters having the form below:

$$\text{- } \mathbf{p}_1 = \left(\begin{array}{c|ccc} rank & \begin{array}{c} p_1^i \\ 1 \leq i \leq q-1 \end{array} & p_1^q & \begin{array}{c} p_1^i \\ q+1 \leq i \leq I \end{array} \\ \hline 1 & a_1 & a_1 & b_1 \\ \ldots & \ldots & \ldots & \ldots \\ r_1^\delta & b_1 & c_1 & c_1 \\ \ldots & \ldots & \ldots & \ldots \end{array} \right)$$

$$\text{- } \forall m \neq 1,\ \mathbf{p}_m = \left(\begin{array}{c|ccc} rank & \begin{array}{c} p_m^i \\ 1 \leq i \leq q-1 \end{array} & p_m^q & \begin{array}{c} p_m^i \\ q+1 \leq i \leq I \end{array} \\ \hline 1 & a_m & b_m & a_m \\ \ldots & \ldots & \ldots & \ldots \end{array} \right).$$

Clearly, $\prod_{m \in \mathcal{M}} F_I(\mathbf{p}_m) = \{\mathbf{a}\}$. If $\boldsymbol{\delta} = (\overbrace{\delta, \ldots, \delta}^{I})$, the outcome profile looks like below:

$$\text{- } \boldsymbol{\delta}(\mathbf{p}) = \left(\begin{array}{c|ccc} rank & \begin{array}{c} \delta(\mathbf{p}^i) \\ 1 \leq i \leq q-1 \end{array} & \delta(\mathbf{p}^q) & \begin{array}{c} \delta(\mathbf{p}^i) \\ q+1 \leq i \leq I \end{array} \\ \hline 1 & \mathbf{a} & (a_1, \mathbf{b}_{-1}) & (b_1, \mathbf{a}_{-1}) \\ 2 & (b_1, \mathbf{a}_{-1}) & (c_1, \mathbf{b}_{-1}) & (c_1, \mathbf{a}_{-1}) \\ \ldots & \ldots & \ldots & \ldots \end{array} \right),$$

where $\mathbf{a} = (a_1, \ldots, a_M)$ and $\mathbf{b} = (b_1, \ldots, b_M)$.

Since $F_q[\boldsymbol{\delta}(\mathbf{p})] = \{(b_1, \mathbf{a}_{-1})\}$, we conclude that $[\prod_{m \in \mathcal{M}} F_q(\mathbf{p}_m)] \cap F_q[\boldsymbol{\delta}(\mathbf{p})] = \emptyset$. Hence, F_q fails weak product stability at δ.

Case 2: $\widetilde{R}^\delta \neq (r_1^\delta, \overbrace{1, \ldots, 1}^{M-1})$.

Define $\mathcal{M}_1 = \{m > 1 : r_m^\delta = 1\}$, and $\mathcal{M}_2 = \{m > 1 : r_m^\delta \neq 1\}$. Pick a coordinate profile $\mathbf{p}$ having the form below:

$$\text{- } \mathbf{p}_1 = \left(\begin{array}{c|ccc} rank & \begin{array}{c} p_1^i \\ 1 \leq i \leq q-1 \end{array} & p_1^q & \begin{array}{c} p_1^i \\ q+1 \leq i \leq I \end{array} \\ \hline 1 & a_1 & a_1 & b_1 \\ \ldots & \ldots & \ldots & \ldots \\ r_m^\delta & b_1 & b_1 & c_1 \\ \ldots & \ldots & \ldots & \ldots \end{array} \right),$$

- $\forall m \in \mathcal{M}_1$, $\mathbf{p}_m = \left(\begin{array}{c|ccc} rank & \begin{array}{c} p_m^i \\ 1 \leq i \leq q-1 \end{array} & p_m^q & \begin{array}{c} p_m^i \\ q+1 \leq i \leq I \end{array} \\ \hline 1 & a_m & a_m & a_m \\ \dots & \dots & \dots & \dots \end{array}\right)$, and

- $\forall m \in \mathcal{M}_2$, $\mathbf{p}_m = \left(\begin{array}{c|ccc} rank & \begin{array}{c} p_m^i \\ 1 \leq i \leq q-1 \end{array} & p_m^q & \begin{array}{c} p_m^i \\ q+1 \leq i \leq I \end{array} \\ \hline 1 & a_m & c_m & b_m \\ \dots & \dots & \dots & \dots \\ r_m^\delta & b_m & a_m & c_m \\ \dots & \dots & \dots & \dots \end{array}\right)$.

Check that an outcome $\overrightarrow{d} = (d_1, \dots, d_M) \in \mathcal{A}$ belongs to $\prod_{m \in \mathcal{M}} F_q(\mathbf{p}_m)$ only if $d_m = a_m$ for all $m \in \mathcal{M}_1 \cup \{1\}$. If $\boldsymbol{\delta} = (\overbrace{\delta, \dots, \delta}^{I})$, the outcome profile looks like below:

$$\boldsymbol{\delta}(\mathbf{p}) = \left(\begin{array}{c|ccc} rank & \begin{array}{c} \delta(\mathbf{p}^i) \\ 1 \leq i \leq q-1 \end{array} & \delta(\mathbf{p}^q) & \begin{array}{c} \delta(\mathbf{p}^i) \\ q+1 \leq i \leq I \end{array} \\ \hline 1 & \mathbf{a} & (\mathbf{a}_{-\mathcal{M}_2}, \mathbf{c}_{\mathcal{M}_2}) & (\mathbf{a}_{\mathcal{M}_1}, \mathbf{b}_{-\mathcal{M}_1}) \\ 2 & (\mathbf{a}_{\mathcal{M}_1}, \mathbf{b}_{-\mathcal{M}_1}) & (b_1, \mathbf{a}_{-1}) & (\mathbf{a}_{\mathcal{M}_1}, \mathbf{c}_{-\mathcal{M}_1}) \\ \dots & \dots & \dots & \dots \end{array}\right),$$

where $\mathbf{a} = (a_1, \dots, a_M)$, $\mathbf{b} = (b_1, \dots, b_M)$ and $\mathbf{c} = (c_1, \dots, c_M)$. Note that $\widetilde{R}^\delta \neq (r_1^\delta, \overbrace{1, \dots, 1}^{M-1})$ implies $\mathcal{M}_2 \neq \emptyset$. Finally, since $F_q[\boldsymbol{\delta}(\mathbf{p})] = \{(\mathbf{a}_{\mathcal{M}_1}, \mathbf{b}_{-\mathcal{M}_1})\}$, we conclude that $[\prod_{m \in \mathcal{M}} F_q(\mathbf{p}_m)] \cap F_q[\boldsymbol{\delta}(\mathbf{p})] = \emptyset$. Hence, F_q fails weak product stability at δ.

Next, we consider the case where $q = \frac{I}{2}$, with I being even.

Proposition 4 *Take any even $I \geq 2$.*
(1) If $I \geq 4$, $F_{\frac{I}{2}}$ is weakly product stable at no preference δ.
(2) If $I = 2$, F_1 is weakly product stable at a preference δ if and only if δ ranks $\overrightarrow{1}$ first.

Proof The proof of assertion (1) is very similar to the proof of proposition 3. Pick $I = 2q$ with $q \geq 2$, and let δ be a preference at which $F_{\frac{I}{2}}$ is weakly product stable. By lemma 3, δ must rank $\overrightarrow{1}$ first. Denote by $\widetilde{R}^\delta = (r_1^\delta, \dots, r_M^\delta)$ the second-best rank vector for δ. Since $\widetilde{R}^\delta \neq \overrightarrow{1}$, we can assume w.l.o.g. that $r_1^\delta > 1$.

Case 1: $\widetilde{R}^\delta = (r_1^\delta, \overbrace{1, \dots, 1}^{M-1})$.
Take a coordinate profile $\mathbf{p}$ with even $I \geq 4$ voters having the form below:

- $\mathbf{p}_1 = \left(\begin{array}{c|cccc} rank & \begin{array}{c} p_1^i \\ 1 \leq i \leq \frac{I}{2}-1 \end{array} & p_1^{\frac{I}{2}} & p_1^{\frac{I}{2}+1} & \begin{array}{c} p_1^i \\ \frac{I}{2}+2 \leq i \leq I \end{array} \\ \hline 1 & a_1 & a_1 & c_1 & b_1 \\ \dots & \dots & \dots & \dots & \dots \\ r_1^\delta & b_1 & b_1 & b_1 & c_1 \\ \dots & \dots & \dots & \dots & \dots \end{array}\right)$

- $\forall m \neq 1$, $\mathbf{p}_m = \left(\begin{array}{c|cccc} rank & \begin{array}{c} p_m^i \\ 1 \leq i \leq \frac{I}{2}-1 \end{array} & p_m^{\frac{I}{2}} & p_m^{\frac{I}{2}+1} & \begin{array}{c} p_m^i \\ \frac{I}{2}+2 \leq i \leq I \end{array} \\ \hline 1 & a_m & b_m & a_m & a_m \\ \dots & \dots & \dots & \dots & \dots \end{array}\right)$.

Let $\mathbf{a} = (a_1, \ldots, a_M)$ and $\mathbf{b} = (b_1, \ldots, b_M)$. Clearly, $\prod_{m \in \mathcal{M}} F_I(\mathbf{p}_m) = \{\mathbf{a}\}$. If $\boldsymbol{\delta} = (\overbrace{\delta, \ldots, \delta}^{I})$, the outcome profile looks like below:

$$
\text{- } \boldsymbol{\delta}(\mathbf{p}) = \left(\begin{array}{c|cccc}
rank & \begin{array}{c}\delta(\mathbf{p}^i) \\ 1 \le i \le \frac{I}{2} - 1\end{array} & \delta(\mathbf{p}^{\frac{I}{2}}) & \delta(\mathbf{p}^{\frac{I}{2}+1}) & \begin{array}{c}\delta(\mathbf{p}^i) \\ \frac{I}{2} + 2 \le i \le I\end{array} \\
\hline
1 & \mathbf{a} & (a_1, \mathbf{b}_{-1}) & (c_1, \mathbf{a}_{-1}) & (b_1, \mathbf{a}_{-1}) \\
2 & (b_1, \mathbf{a}_{-1}) & \mathbf{b} & (b_1, \mathbf{a}_{-1}) & (c_1, \mathbf{a}_{-1}) \\
\ldots & \ldots & \ldots & \ldots & \ldots
\end{array} \right).
$$

Since $F_{\frac{I}{2}}[\boldsymbol{\delta}(\mathbf{p})] = \{(b_1, \mathbf{a}_{-1})\}$, we conclude that $[\prod_{m \in \mathcal{M}} F_{\frac{I}{2}}(\mathbf{p}_m)] \cap F_{\frac{I}{2}}[\boldsymbol{\delta}(\mathbf{p})] = \emptyset$. Hence, $F_{\frac{I}{2}}$ fails weak product stability at δ.

Case 2: $\widetilde{R}^{\delta} \neq (r_1^{\delta}, \overbrace{1, \ldots, 1}^{M-1})$.

Define $\mathcal{M}_1 = \{m > 1 : r_m^{\delta} = 1\}$, and $\mathcal{M}_2 = \{m > 1 : r_m^{\delta} \neq 1\}$. Pick a coordinate profile $\mathbf{p}$ having the form below:

$$
\text{- } \mathbf{p}_1 = \left(\begin{array}{c|cccc}
rank & \begin{array}{c}p_1^i \\ 1 \le i \le \frac{I}{2} - 1\end{array} & p_1^{\frac{I}{2}} & p_1^{\frac{I}{2}+1} & \begin{array}{c}p_1^i \\ \frac{I}{2} + 2 \le i \le I\end{array} \\
\hline
1 & a_1 & a_1 & c_1 & b_1 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
r_1^{\delta} & b_1 & b_1 & b_1 & c_1 \\
\ldots & \ldots & \ldots & \ldots & \ldots
\end{array} \right),
$$

$$
\text{- } \forall m \in \mathcal{M}_1,\ \mathbf{p}_m = \left(\begin{array}{c|cccc}
rank & \begin{array}{c}p_m^i \\ 1 \le i \le \frac{I}{2} - 1\end{array} & p_m^{\frac{I}{2}} & p_m^{\frac{I}{2}+1} & \begin{array}{c}p_m^i \\ \frac{I}{2} + 2 \le i \le I\end{array} \\
\hline
1 & a_m & a_m & a_m & a_m \\
\ldots & \ldots & \ldots & \ldots & \ldots
\end{array} \right),
$$

$$
\text{- } \forall m \in \mathcal{M}_2,\ \mathbf{p}_m = \left(\begin{array}{c|cccc}
rank & \begin{array}{c}p_m^i \\ 1 \le i \le \frac{I}{2} - 1\end{array} & p_m^{\frac{I}{2}} & p_m^{\frac{I}{2}+1} & \begin{array}{c}p_m^i \\ \frac{I}{2} + 2 \le i \le I\end{array} \\
\hline
1 & a_m & c_m & a_m & b_m \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
r_1^{\delta} & b_m & b_m & b_m & c_m \\
\ldots & \ldots & \ldots & \ldots & \ldots
\end{array} \right).
$$

Let $\mathbf{a} = (a_1, \ldots, a_M)$, $\mathbf{b} = (b_1, \ldots, b_M)$, and $\mathbf{c} = (c_1, \ldots, c_M)$. Clearly, $\prod_{m \in \mathcal{M}} F_{\frac{I}{2}}(\mathbf{p}_m) = \{\mathbf{a}\}$. If $\boldsymbol{\delta} = (\overbrace{\delta, \ldots, \delta}^{I})$, the outcome profile looks like below:

$$
\text{- } \boldsymbol{\delta}(\mathbf{p}) = \left(\begin{array}{c|cccc}
rank & \begin{array}{c}\delta(\mathbf{p}^i) \\ 1 \le i \le \frac{I}{2} - 1\end{array} & \delta(\mathbf{p}^{\frac{I}{2}}) & \delta(\mathbf{p}^{\frac{I}{2}+1}) & \begin{array}{c}\delta(\mathbf{p}^i) \\ \frac{I}{2} + 2 \le i \le I\end{array} \\
\hline
1 & \mathbf{a} & (\mathbf{a}_{-\mathcal{M}_2}, \mathbf{c}_{\mathcal{M}_2}) & (c_1, \mathbf{a}_{-1}) & (\mathbf{a}_{\mathcal{M}_1}, \mathbf{b}_{-\mathcal{M}_1}) \\
2 & (\mathbf{a}_{\mathcal{M}_1}, \mathbf{b}_{-\mathcal{M}_1}) & (\mathbf{a}_{\mathcal{M}_1}, \mathbf{b}_{-\mathcal{M}_1}) & (b_1, \mathbf{a}_{-1}) & (\mathbf{a}_{\mathcal{M}_1}, \mathbf{c}_{-\mathcal{M}_1}) \\
\ldots & \ldots & \ldots & \ldots & \ldots
\end{array} \right).
$$

Note that $\widetilde{R}^{\delta} \neq (r_1^{\delta}, \overbrace{1, \ldots, 1}^{M-1})$ implies $\mathcal{M}_2 \neq \emptyset$. Since $F_{\frac{I}{2}}[\boldsymbol{\delta}(\mathbf{p})] = \{(\mathbf{a}_{\mathcal{M}_1}, \mathbf{b}_{-\mathcal{M}_1})\}$, we conclude that $[\prod_{m \in \mathcal{M}} F_{\frac{I}{2}}(\mathbf{p}_m)] \cap F_{\frac{I}{2}}[\boldsymbol{\delta}(\mathbf{p})] = \emptyset$. Hence, $F_{\frac{I}{2}}$ fails weak product stability at δ. This completes the proof of assertion (1).

To show assertion (2), observe that the necessity part comes from lemma 3. To show the sufficiency part, pick any coordinate profile $\mathbf{p}$ and any neutral preference extension δ that ranks $\overrightarrow{1}$ first. Take any voter i and take $\overrightarrow{a} = (a_1, \ldots, a_m) \in \mathcal{A}$ such that $R(\overrightarrow{a}, p^i) = \overrightarrow{1}$. By definition of F_1, $\overrightarrow{a} \in \prod_{m\in\mathcal{M}} F_1(\mathbf{p}_m)$. Since δ that ranks $\overrightarrow{1}$ first, then $\overrightarrow{a} \in F_1[\boldsymbol{\delta}(\mathbf{p})]$ for all $\boldsymbol{\delta}$ with $\delta^i = \delta$. Thus $[\prod_{m\in\mathcal{M}} F_1(\mathbf{p}_m)] \cap F_1[\boldsymbol{\delta}(\mathbf{p})] \neq \emptyset$ and the conclusion follows.

We turn to $q-$approval rules with $1 \leq q < \frac{I}{2}$. Note that this condition implies $I > 2$.

Proposition 5 *(1) If $I \geq 4$ and $1 \leq q < \frac{I}{2}$, F_q is weakly product stable at no preference δ.*
(2) If $I = 3$ and $M \geq 3$, F_q is weakly product stable at no preference δ.
(3) If $I = 3$ and $M = 2$, F_1 is weakly product stable at preference δ if and only if δ ranks $\overrightarrow{1}$ first.

Proof To show assertion (1), take $q \in \{1, \ldots, \frac{I}{2}\}$ with $I \geq 4$, and pick a preference δ at which F_q is weakly product stable. By lemma 3, δ must rank $\mathbf{1}$ at top.

Suppose first that $I \geq 5$.

Consider the coordinate profile $\mathbf{p}$ below:

- $\mathbf{p}_1 = \left(\begin{array}{c|cccc} \mathit{rank} & 1, \ldots, \lceil \frac{I}{2} \rceil - 1 & \lceil \frac{I}{2} \rceil & \lceil \frac{I}{2} \rceil + 1, \ldots, I-1 & I \\ \hline 1 & a_1 & a_1 & b_1 & c_1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{array}\right)$,
- $\forall m \neq 1$, $\mathbf{p}_m = \left(\begin{array}{c|ccc} \mathit{rank} & 1, \ldots, \lceil \frac{I}{2} \rceil - 1 & \lceil \frac{I}{2} \rceil & \lceil \frac{I}{2} \rceil + 1, \ldots, I \\ \hline 1 & b_m & a_m & a_m \\ \ldots & \ldots & \ldots & \ldots \end{array}\right)$.

Check that $\prod_{m\in\mathcal{M}} F_q(\mathbf{p}_m) = \{\mathbf{a}\}$, where $\mathbf{a} = (a_1, \ldots, a_M)$. If $\boldsymbol{\delta} = (\overbrace{\delta, \ldots, \delta}^{I})$, one gets an extended profile such as below:

$$\boldsymbol{\delta}(\mathbf{p}) = \left(\begin{array}{c|cccc} \mathit{rank} & 1, \ldots, \lceil \frac{I}{2} \rceil - 1 & \lceil \frac{I}{2} \rceil & \lceil \frac{I}{2} \rceil + 1, \ldots, I-1 & I \\ \hline 1 & (a_1, \mathbf{b}_{-1}) & \mathbf{a} & (b_1, \mathbf{a}_{-1}) & (c_1, \mathbf{a}_{-1}) \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{array}\right),$$

where $\mathbf{b} = (b_1, \ldots, b_M) \in \mathcal{A}$. Check that $F_q[\boldsymbol{\delta}(\mathbf{p})] \subseteq \{(a_1, \mathbf{b}_{-1}), (b_1, \mathbf{a}_{-1})\}$.[12] Thus, $[\prod_{m\in\mathcal{M}} F_q(\mathbf{p}_m)] \cap F_q[\boldsymbol{\delta}(\mathbf{p})] = \emptyset$, which shows that F_q fails weak product stability at δ.

Now, suppose that $I = 4$. Hence, $q = 1$.

Consider the coordinate profile $\mathbf{p}$ below:

- $\mathbf{p}_1 = \left(\begin{array}{c|cccc} \mathit{rank} & 1 & 2 & 3 & 4 \\ \hline 1 & a_1 & a_1 & b_1 & c_1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{array}\right)$, $\mathbf{p}_2 = \left(\begin{array}{c|cccc} \mathit{rank} & 1 & 2 & 3 & 4 \\ \hline 1 & b_2 & c_2 & a_2 & a_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{array}\right)$, and
- $\forall m \neq 1, 2$, $\mathbf{p}_m = \left(\begin{array}{c|cccc} \mathit{rank} & 1 & 2 & 3 & 4 \\ \hline 1 & a_m & a_m & a_m & a_m \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{array}\right)$.

Check that $\prod_{m\in\mathcal{M}} F_1(\mathbf{p}_m) = \{\mathbf{a}\}$. If $\boldsymbol{\delta} = (\delta, \delta, \delta, \delta)$ one gets an extended profile as below:

$$\boldsymbol{\delta}(\mathbf{p}) = \left(\begin{array}{c|cccc} \mathit{rank} & 1 & 2 & 3 & 4 \\ \hline 1 & (b_2, \mathbf{a}_{-2}) & (c_2, \mathbf{a}_{-2}) & (b_1, \mathbf{a}_{-1}) & (c_1, \mathbf{a}_{-1}) \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{array}\right).$$

Clearly, $F_1[\boldsymbol{\delta}(\mathbf{p})] = \{(b_2, \mathbf{a}_{-2}), (c_2, \mathbf{a}_{-2}), (b_1, \mathbf{a}_{-1}), (c_1, \mathbf{a}_{-1})\}$. Since $[\prod_{m\in\mathcal{M}} F_1(\mathbf{p}_m)] \cap F_1[\boldsymbol{\delta}(\mathbf{p})] = \emptyset$, F_q is not weakly product stable at δ, which completes the proof of assertion (1).

To show assertion (2), take $M \geq 3$ and pick a coordinate profile $\mathbf{p}$ such as below:

[12] The reader will check that $F_q[\boldsymbol{\delta}(\mathbf{p})] = \{(a_1, \mathbf{b}_{-1}), (b_1, \mathbf{a}_{-1})\}$ if I is even and $F_q[\boldsymbol{\delta}(\mathbf{p})] = \{(a_1, \mathbf{b}_{-1})\}$ if I is odd.

- $\mathbf{p}_1 = \left(\begin{array}{c|ccc} rank & 1 & 2 & 3 \\ \hline 1 & b_1 & a_1 & a_1 \\ \dots & \dots & \dots & \dots \end{array}\right)$, $\mathbf{p}_2 = \left(\begin{array}{c|ccc} rank & 1 & 2 & 3 \\ \hline 1 & a_2 & b_2 & a_2 \\ \dots & \dots & \dots & \dots \end{array}\right)$,
- $\mathbf{p}_3 = \left(\begin{array}{c|ccc} rank & 1 & 2 & 3 \\ \hline 1 & a_3 & a_3 & b_3 \\ \dots & \dots & \dots & \dots \end{array}\right)$, and
- $\mathbf{p}_m = \left(\begin{array}{c|ccc} rank & 1 & 2 & 3 \\ \hline 1 & a_m & a_m & a_m \\ \dots & \dots & \dots & \dots \end{array}\right)$ for $m \neq 1, 2, 3$.

Check that $\prod_{m\in\mathcal{M}} F_1(\mathbf{p}_m) = \{\mathbf{a}\}$, where $\mathbf{a} = (a_1, \dots, a_M)$. Now, pick a preference δ that ranks $\overrightarrow{1}$ first. If $\boldsymbol{\delta} = (\delta, \delta, \delta)$ one gets an extended profile as below:

- $\boldsymbol{\delta}(\mathbf{p}) = \left(\begin{array}{c|ccc} rank & 1 & 2 & 3 \\ \hline 1 & (b_1, \mathbf{a}_{-1}) & (b_2, \mathbf{a}_{-2}) & (b_3, \mathbf{a}_{-3}) \\ \dots & \dots & \dots & \dots \end{array}\right)$.

As $F_1[\boldsymbol{\delta}(\mathbf{p})] = \{(b_1, \mathbf{a}_{-1}), (b_2, \mathbf{a}_{-2}), (b_3, \mathbf{a}_{-3})\}$, we get $[\prod_{m\in\mathcal{M}} F_1(\mathbf{p}_m)] \cap F_1[\boldsymbol{\delta}(\mathbf{p})] = \emptyset$. Therefore F_1 is not weakly product stable at δ, and by lemma 3, the proof of assertion (2) is complete.

Finally, we show assertion (3). As for proposition 4, the necessity part results from lemma 3. To show the sufficiency part, pick any coordinate profile $\mathbf{p}$. For each $m \in \{1, 2\}$ and for each $i \in \mathcal{I}$, define $a^i_m \in \mathcal{A}_m$ as the top element of $\mathcal{A}_m$ for p^i_m. Suppose first that $\forall m \in \{1, 2\}$, $\forall i, j \in \mathcal{I}$ with $i \neq j$, we have $a^i_m \neq a^j_m$. By definition of F_1, $\forall i \in \mathcal{I}$, $(a^i_1, a^i_2) \in F_1(\mathbf{p}_1) \times F_1(\mathbf{p}_2)$ and $R[(a^i_1, a^i_2), p^i] = \overrightarrow{1}$. If δ is a preference that ranks $\overrightarrow{1}$ first, we get $\forall i \in \mathcal{I}$, $\delta(p^i)[(a^i_1, a^i_2)] = 1$, and therefore $(a^i_1, a^i_2) \in F_1[\boldsymbol{\delta}(\mathbf{p})]$ where $\boldsymbol{\delta} = (\delta, \delta, \delta)$. Hence, $[F_1(\mathbf{p}_1) \times F_1(\mathbf{p}_2)] \cap F_1[\boldsymbol{\delta}(\mathbf{p})] \neq \emptyset$, which shows that F_1 is weakly product stable at δ. Now, suppose that $\exists m \in \{1, 2\}$, $\exists i, j \in \mathcal{I}$ with $i \neq j$ and $a^i_m = a^j_m = a_m$. We can assume w.l.o.g. that $m = 2$, $i = 1$ and $j = 2$. If $\exists i', j' \in \mathcal{I}$ with $i' \neq j'$ and $a^{i'}_1 = a^{j'}_1 = a_1$, then $F_1(\mathbf{p}_1) \times F_1(\mathbf{p}_2) = \{(a_1, a_2)\}$. Moreover, either $\{1, 2\} = \{i', j'\}$ or $\{1, 2\} \cap \{i', j'\} = \{1\}$. In the former case, we get $\delta(p^i)[(a_1, a_2)] = \delta(p^j)[(a_1, a_2)] = 1$, and therefore $F_1[\boldsymbol{\delta}(\mathbf{p})] = \{(a_1, a_2)\} = F_1(\mathbf{p}_1) \times F_1(\mathbf{p}_2)$. In the latter case, we get $F_1[\boldsymbol{\delta}(\mathbf{p})] = \{(a_1, a_2), (a^2_1, a_2), (a_1, a^3_2)\}$. Hence, $[F_1(\mathbf{p}_1) \times F_1(\mathbf{p}_2)] \subseteq F_1[\boldsymbol{\delta}(\mathbf{p})]$. Finally, if $a^i_1 \neq a^j_1$ for all $i, j \in \mathcal{I}$ with $i \neq j$, then $F_1(\mathbf{p}_1) \times F_1(\mathbf{p}_2) = \{(a^1_1, a_2), (a^2_1, a_2), (a^3_1, a^3_2)\} = F_1[\boldsymbol{\delta}(\mathbf{p})]$. Hence, F_1 is weakly product stable at δ in all possible configurations.[13]

Propositions 3, 4, and 5 are summarized below.

Theorem 3 *If either $[I \geq 3$ and $\frac{I}{2} < q < I]$ or $[I \geq 4$ and $1 \leq q \leq \frac{I}{2}]$ or $[I = 3$ and $q = 1$, and $M \geq 3]$, F_q is weakly product stable at no preference. Moreover, if either $[I = 3$ and $M = 2]$ or $I = 2]$, F_1 is weakly product stable at preference δ if and only if δ ranks $\overrightarrow{1}$ first.*

7 Concluding comments

We introduce the concept of (weak) product stability for a voting rule in the context of combinatorial vote. Product stability holds if at every profile of coordinate-wise preferences and preferences over outcomes, every sequential outcome, defined as an outcome formed by coordinate-wise chosen alternatives (based on coordinate-wise preferences), is also a direct outcome, that is, an outcome chosen when the voting rule

[13] Observe that product stability prevails in all cases but the one where the 3 voters disagree on their first-best element in each of the coordinates.

is applied to preferences over outcomes. Weak product stability holds if at least one sequential outcome is a direct outcome. It is already known that no desirable (either Pareto efficient or neutral) voting rule is product stable unless very specific conditions are retained about the dimension of the outcome space (Benoît and Kornhauser, 2010). Hence, unless discarding appealing properties, voting by parts may not yield an outcome that would arise if voting all-at-once for the whole.

A natural question is whether the failure of product stability is rare or frequent. A possible approach to this question is of computational nature, which consists of calculating for a given voting rule the frequency of failure according to certain probabilistic assumptions upon the distribution of profiles. Another approach, which we follow here, is to identify restrictions upon preferences that restore product stability. The size of the largest preference domain that satisfies such a restriction may be seen as a proxy for the degree of product instability of a voting rule. This paper characterizes the largest neutral preference domains for which the Fallback Bargaining rule is weakly product stable. It also shows that no neutral domain makes a $q-$approval voting rule product stable, where q is any number between 1 and the number I of voters. Moreover, if q is strictly less than I, no neutral domain makes a $q-$approval voting rule even weakly product stable, unless very special cases are considered.

Our paper relates to the one by Aslan et al. (2021), who consider Condorcet voting rules and neutral preference domains where coordinate-wise preferences ensure the existence of a Condorcet winner in each coordinate. They prove that, in restriction to these domains, a Condorcet voting rule is product stable if and only if all voters have preferences over outcomes that are lexicographic w.r.t. a common ordering of coordinates. Other studies investigating preference domains that ensure some consistency between sequential and direct outcomes in the context of multiple referendum are due to Laffond and Lainé (2006) and Cuhadaroglu and Lainé (2012).

Preference domains that ensure product stability remains to be investigated for other voting rules (e.g., positional rules). Exploring the logical relation between the properties of voting rules and the structure of preference domains over which they are product stable opens the route to a large road of further research.

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9 Appendix: Proof of theorem 1

9.1 Proof of sufficiency part

We want to show that for any $\widetilde{q} \in \mathcal{L}_{\mathcal{M}}$, F_I is weakly product stable at $\delta^{\widetilde{q}}$.

Take any $\widetilde{q} \in \mathcal{L}_{\mathcal{M}}$. Up to a reshuffling of coordinates $m \in \mathcal{M}$, one can assume w.l.o.g. that $\widetilde{q} = [12 \dots M]$. Suppose that there exists a coordinate profile $\mathbf{p}$ such that $[\prod_{m \in \mathcal{M}} F_I(\mathbf{p}_m)] \cap F_I[\boldsymbol{\delta}(\mathbf{p})] = \emptyset$, where $\boldsymbol{\delta} = (\delta^{\widetilde{q}}, \dots, \delta^{\widetilde{q}})$. Take any $\overrightarrow{b} = (b_1, \dots, b_M) \in \prod_{m \in \mathcal{M}} F_I(\mathbf{p}_m)$ and $\overrightarrow{a} = (a_1, \dots, a_M) \in F_I[\boldsymbol{\delta}(\mathbf{p})]$. Define $m_1 \in \mathcal{M}$ as the coordinate with the lowest index such that $a_{m_1} \neq b_{m_1}$. Hence, we can write $\overrightarrow{a} = (\overrightarrow{b}_{<m_1}, \overrightarrow{a}_{\geq m_1})$. Moreover, define r^* as the $I-$value of $\overrightarrow{b}$ at $\boldsymbol{\delta}(\mathbf{p})$. By definition of the $I-$value, r^* is the highest rank given to $\overrightarrow{b}$ by some linear order in $\boldsymbol{\delta}(\mathbf{p})$. Pick $i^* \in \mathcal{I}$ for whom $\delta^{\widetilde{q}}(p^{i^*})(\overrightarrow{b}) = r^*$. Since $p^{i^*}_{m_1} \in \mathcal{L}_{\mathcal{A}_{m_1}}$ and $a_{m_1} \neq b_{m_1}$, we have $p^{i^*}_{m_1}(b_{m_1}) \neq p^{i^*}_{m_1}(a_{m_1})$. Moreover, since $\overrightarrow{a} \in F_I[\boldsymbol{\delta}(\mathbf{p})]$ and whereas $\overrightarrow{b} \notin F_I[\boldsymbol{\delta}(\mathbf{p})]$, $V_I[\overrightarrow{a}, \boldsymbol{\delta}(\mathbf{p})]$, that is the $I-$value of $\overrightarrow{a}$ in $\boldsymbol{\delta}(\mathbf{p})$ is strictly less than r^*. It follows from the

definition of $\delta^{\widetilde{q}}$ that $p^{i^*}_{m_1}(a_{m_1}) < p^{i^*}_{m_1}(b_{m_1})$. Now, pick any $i \neq i^*$ and suppose that $p^{i}_{m_1}(a_{m_1}) > p^{i^*}_{m_1}(b_{m_1})$. By the definition of $\delta^{\widetilde{q}}$, we get $\delta^{\widetilde{q}}(p^i)(\overrightarrow{a}) > \delta^{\widetilde{q}}(p^{i^*})(\overrightarrow{b}) = r^*$. As this would imply $V_I[\overrightarrow{a}, \boldsymbol{\delta}(\mathbf{p})] > r^*$, we contradict $\overrightarrow{a} \in F_I[\boldsymbol{\delta}(\mathbf{p})]$. Hence, we have shown that (1) $p^{i^*}_{m_1}(a_{m_1}) < p^{i^*}_{m_1}(b_{m_1})$ and (2) $p^{i}_{m_1}(a_{m_1}) \leq p^{i^*}_{m_1}(b_{m_1})$ for all $i \in \mathcal{I} \setminus \{i^*\}$. Observe that $p^{i}_{m_1}(a_{m_1}) < p^{i^*}_{m_1}(b_{m_1})$ for all $i \in \mathcal{I}$ contradicts with $b_{m_1} \in F_I(\mathbf{p}_{m_1})$. Thus, $p^{i}_{m_1}(a_{m_1}) = p^{i^*}_{m_1}(b_{m_1})$ for some $i \in \mathcal{I}$, which in turn implies $a_{m_1} \in F_I(\mathbf{p}_{m_1})$.

Next consider $\overrightarrow{b}^{(1)} = (b^{(1)}_1, \ldots, b^{(1)}_M) = (a_{m_1}, \overrightarrow{b}_{-m_1})$. Since $\overrightarrow{b} \in \prod_{m\in\mathcal{M}} F_I(\mathbf{p}_m)$ and $a_{m_1} \in F_I(\mathbf{p}_{m_1})$, then $\overrightarrow{b}^{(1)} \in \prod_{m\in\mathcal{M}} F_I(\mathbf{p}_m)$. All is done if $\overrightarrow{b}^{(1)} \in F_I[\boldsymbol{\delta}(\mathbf{p})]$. Suppose that $\overrightarrow{b}^{(1)} \notin F_I[\boldsymbol{\delta}(\mathbf{p})]$. Consider again $\overrightarrow{a}$ and define $m_2 \in \mathcal{M}$ as the coordinate with the lowest index $\overrightarrow{a}$ and $\overrightarrow{b}^{(1)}$ disagree upon. By construction, we have $m_2 > m_1$. Hence, we can write $\overrightarrow{a} = (\overrightarrow{b}^{(1)}_{<m_2}, \overrightarrow{a}_{\geq m_2}) \in F_I[\boldsymbol{\delta}(\mathbf{p})]$. By applying to m_2 the same argument as the one for m_1, we get $a_{m_2} \in F_I(\mathbf{p}_{m_2})$. This shows that $\overrightarrow{b}^{(2)} = (a_{m_2}, \overrightarrow{b}^{(1)}_{-m_2}) \in \prod_{m\in\mathcal{M}} F_I(\mathbf{p}_m)$. By replicating the same argument, and because of the finiteness of $\mathcal{M}$, there must exist $T \leq M$ such that $\overrightarrow{b}^{(T)} = \overrightarrow{a}$. By construction, we have $\overrightarrow{b}^{(T)} = (a_{m_T}, \overrightarrow{b}^{(T-1)}_{-m_T}) \in \prod_{m\in\mathcal{M}} F_I(\mathbf{p}_m)$. Since $\overrightarrow{a} \in F_I[\boldsymbol{\delta}(\mathbf{p})]$, then $[\prod_{m\in\mathcal{M}} F_I(\mathbf{p}_m)] \cap F_I[\boldsymbol{\delta}(\mathbf{p})] \neq \emptyset$, which shows that F_I is weakly product stable at $\delta^{\widetilde{q}}$.

9.2 Proof of necessary part

The proof of the necessary part is organized in two lemmas. The first one states that the weak product stability of F_I requires all voters to use a lexicographic preference.

Lemma 6 *Let F_I be product stable over $\mathcal{D}$. If $I \geq I^* + 3$, then $\delta \in \mathcal{D}$ only if $\delta = \delta^{\widetilde{q}}$ for some $\widetilde{q} \in \mathcal{L}_{\mathcal{M}}$.*

Proof Let $\mathcal{R}^* = \{R \in \mathcal{R} : R = (2, \overrightarrow{1}_{-m}) \text{ for some } m \in \mathcal{M}\}$. Take any $\delta \in \mathcal{D}$. By Proposition 1, δ is responsive. Write $\delta|_{\mathcal{R}^*}$ as $(2, \mathbf{1}_{-m_M})\ \delta|_{\mathcal{R}^*}\ (2, \mathbf{1}_{-m_{M-1}})\ \delta|_{\mathcal{R}^*} \ldots \delta|_{\mathcal{R}^*}(2, \mathbf{1}_{-m_2})\ \delta|_{\mathcal{R}^*}\ (2, \mathbf{1}_{-m_1})$. Observe that responsiveness implies that δ ranks $(2, \mathbf{1}_{-m_M})$ second in $\mathcal{R}$. Moreover, define $\widetilde{q} \in \mathcal{L}_{\mathcal{M}}$ by $\widetilde{q}^{-1}(n) = m_n$ for all $n \in \mathcal{M}$. For notational simplicity, assume that $\widetilde{q} = [1, 2, \ldots, M]$. Hence, δ is responsive, ranks $\mathbf{1}$ at top, $(2, \mathbf{1}_{-M})$ second and $\delta|_{\mathcal{R}^*} = [(2, \mathbf{1}_{-M}), (2, \mathbf{1}_{-(M-1)}), \ldots, (2, \mathbf{1}_{-1})]$.

Suppose, towards a contradiction, that $\delta \neq \delta^{\widetilde{q}}$. Thus, there exist $m^* \in \mathcal{M}$ and $R, R' \in \mathcal{R}$ with $R = (R_{<m^*}, r_{m^*}, r_{m^*+1}, \ldots, r_M)$, where $R_{<m^*} = (\overbrace{r_1, \ldots, r_{m^*-1}}^{m^*-1})$, $R' = (R_{<m^*}, r'_{m^*}, r'_{m^*+1}, \ldots, r'_M)$ with $r_{m^*} > r'_{m^*}$, and $R\ \delta\ R'$.

Observe that, by the responsiveness of δ**,** we must have $m^* < M$. Again by the responsiveness of δ**,** one can assume w.l.o.g. that $r_m = 1$ and $r'_m = A_m$ for all $m \in \{m^* + 1, \ldots, M\}$. Thus,

- $R = (R_{<m^*}, r_{m^*}, 1, \ldots, 1)$, where $R_{<m^*} = (\overbrace{r_1, \ldots, r_{m^*-1}}^{m^*-1})$,
- $R' = (R_{<m^*}, r'_{m^*}, A_{m^*+1}, \ldots, A_M)$ with $r_{m^*} > r'_{m^*}$,
- $R\ \delta\ R'$.

We distinguish between two cases.

*Case 1***:** $r_{m^*} < A_{m^*}$.
Define
- $\tilde{R} = (R_{<m^*}, r_{m^*} + 1, 1, \ldots, 1)$,
- $\tilde{R}' = (R_{<m^*}, r'_{m^*} + 1, A_{m^*+1}, \ldots, A_M)$.

*Case 1a***:** $\tilde{R}\ \delta\ \tilde{R}'$.
Consider a coordinate 4-voter profile $\mathbf{p}$ having the general form below:

- $\forall m < m^*$, $\mathbf{p}_m = \left(\begin{array}{ccccc} Rank & p_m^1 & p_m^2 & p_m^3 & p_m^4 \\ \hline 1 & \cdot & \cdot & \cdot & a_m \\ r_m = r'_m & a_m & a_m & a_m & \cdot \end{array}\right)$,

- $\mathbf{p}_{m^*} = \left(\begin{array}{ccccc} Rank & p_{m^*}^1 & p_{m^*}^2 & p_{m^*}^3 & p_{m^*}^4 \\ \hline 1 & \cdot & \cdot & \cdot & a_{m^*} \\ r'_{m^*} & a_{m^*} & a_{m^*} & b_{m^*} & \cdot \\ r'_{m^*}+1 & b_{m^*} & b_{m^*} & \cdot & b_{m^*} \\ r_{m^*}+1 & \cdot & \cdot & a_{m^*} & \cdot \end{array}\right)$,

- $\forall m > m^*$, $\mathbf{p}_m = \left(\begin{array}{ccccc} Rank & p_m^1 & p_m^2 & p_m^3 & p_m^4 \\ \hline 1 & \cdot & \cdot & a_m & a_m \\ 2 & a_m & \cdot & \cdot & \cdot \\ A_m & \cdot & a_m & \cdot & \cdot \end{array}\right)$[14].

Let $\overrightarrow{a} = (a_1, \ldots, a_M)$, $\overrightarrow{b} = (b_1, \ldots, b_M)$, $\overrightarrow{c} = (b_{m^*}, \overrightarrow{a}_{-m^*}) \in \mathcal{A}$, $\mathcal{A}' = \{(b_{m^*}, \overrightarrow{\alpha}_{-m^*}) : \overrightarrow{\alpha} \in \mathcal{A}\} \setminus \{\overrightarrow{c}\}$, and consider a coordinate profile $\overline{\mathbf{p}}$ involving $I \geq \prod_{m \in \mathcal{M} \setminus \{m^*\}} A_m$ +3 voters and such that:

(1) $\forall i \in \{1,2,3\}$, $\overline{p}^i = p^i$,

(2) $\forall i \in \{4, \ldots, I\}$, $\overline{p}^i$ agrees with p^4 on the ranks given to a_m for all m, $\overline{p}^i$ agrees with p^4 on the rank given to b_{m^*}, and $\forall c_{m^*} \in \mathcal{A}_{m^*} \setminus \{a_{m^*}, b_{m^*}\}$ there exists $i(c_{m^*}) \in \{4, \ldots, I\}$ such that $r[c_{m^*}, \overline{p}^{i(c_{m^*})}] = A_{m^*}$,

(3) $\forall \overrightarrow{d} \in \mathcal{A}'$, there exists $i(\overrightarrow{d}) \in \{4, \ldots, I\}$ such that $R[\overrightarrow{d}, \overline{p}^{i(\overrightarrow{d})}] = \tilde{R}'$.

As $I^* \geq \prod_{m \in \mathcal{M} \setminus \{m^*\}} A_m$, condition (3) can be satisfied. Moreover, we have $\prod_{m \in \mathcal{M}} F_I(\overline{\mathbf{p}}) \subseteq \mathcal{A}' \cup \{\overrightarrow{c}\}$ and $\overrightarrow{c} \in \prod_{m \in \mathcal{M}} F_I(\overline{\mathbf{p}})$. Table 1 gives the rank vectors of $\overrightarrow{a}$ and $\overrightarrow{c}$ for each coordinate preference $\overline{p}^i$:

Table 1

i	$R(\overrightarrow{a}, \overline{p}^i)$	$R[\overrightarrow{c} = (b_{m^*}, \overrightarrow{a}_{-m^*}), \overline{p}^i]$
1	$(R_{<m^*}, r'_{m^*}, 2, \ldots, 2)$	$(R_{<m^*}, r'_{m^*}+1, 2, ..., 2)$
2	$R' = (R_{<m^*}, r'_{m^*}, A_{m^*+1}, \ldots, A_M)$	$\tilde{R}' = (R_{<m^*}, r'_{m^*}+1, A_{m^*+1}, \ldots, A_M)$
3	$\tilde{R} = (R_{<m^*}, r_{m^*}+1, 1, \ldots, 1)$	$(R_{<m^*}, r'_{m^*}, 1, \ldots, 1)$
≥ 4	$\mathbf{1}$	$(\overbrace{1, \ldots, 1}^{m^*-1}, r'_{m^*}+1, \overbrace{1, \ldots, 1}^{M-m^*})$

Pick $\boldsymbol{\delta} = (\delta, \ldots, \delta)$. Observe that the responsiveness of δ implies $R(\overrightarrow{c}, \overline{p}^i)\ \delta\ R(\overrightarrow{c}, \overline{p}^2)$ for all $i \neq 2$. Hence $V_I[\overrightarrow{c}, \boldsymbol{\delta}(\overline{\mathbf{p}})]$, the $I-$value of $\overrightarrow{c}$ in $\boldsymbol{\delta}(\overline{\mathbf{p}})$, is equal to $r[\overrightarrow{c}, \delta(\overline{p}^2)]$. Similarly, the responsiveness of δ implies $R(\overrightarrow{a}, \overline{p}^i)\ \delta\ R(\overrightarrow{a}, \overline{p}^j)$ for all $i \geq 4$ and all $j < 4$. Moreover, as $A_m > 2$ for all $m \in \mathcal{M}$, $R(\overrightarrow{a}, \overline{p}^1)\ \delta\ R(\overrightarrow{a}, \overline{p}^2)$. Thus, $V_I[\overrightarrow{a}, \boldsymbol{\delta}(\overline{\mathbf{p}})] \in \{r[\overrightarrow{a}, \delta(\overline{p}^2)],\ r[\overrightarrow{a}, \delta(\overline{p}^3)]\}$. By definition of case 1a, we have $\tilde{R}\ \delta\ \tilde{R}'$, and by responsiveness $R'\ \delta\ \tilde{R}'$, which implies $V_I[\overrightarrow{a}, \boldsymbol{\delta}(\overline{\mathbf{p}})] < V_I[\overrightarrow{c}, \boldsymbol{\delta}(\overline{\mathbf{p}})]$. Therefore, $\overrightarrow{c} \notin F_I[\boldsymbol{\delta}(\overline{\mathbf{p}})]$. Finally, pick any $\overrightarrow{d} \in \mathcal{A}'$. By definition of $\overline{\mathbf{p}}$, we have $V_I[\overrightarrow{d}, \boldsymbol{\delta}(\overline{\mathbf{p}})] \geq r[\overrightarrow{d}, \overline{p}^{i(\overrightarrow{d})}] = r[\overrightarrow{c}, \delta(\overline{p}^2)] = V_I[\overrightarrow{c}, \boldsymbol{\delta}(\overline{\mathbf{p}})]$. As $V_I[\overrightarrow{a}, \boldsymbol{\delta}(\overline{\mathbf{p}})] < V_I[\overrightarrow{c}, \boldsymbol{\delta}(\overline{\mathbf{p}})]$, we get $[\prod_{m \in \mathcal{M}} F_I(\mathbf{p})] \cap F_I[\boldsymbol{\delta}(\overline{\mathbf{p}})] = \emptyset$, in contradiction with the assumption that F_I is weakly stable over $\mathcal{D}$.

Case 1b: $\tilde{R}'\ \delta\ \tilde{R}$.

We use an argument similar to the one for case 1a. Consider the coordinate 4-voter profile $\mathbf{p}$ with the form below:

- $\forall m < m^*$, $\mathbf{p}_m = \left(\begin{array}{c|cccc} Rank & p_m^1 & p_m^2 & p_m^3 & p_m^4 \\ \hline 1 & \cdot & \cdot & \cdot & a_m \\ r_m = r'_m & a_m & a_m & a_m & \cdot \end{array}\right)$,

[14] It is easy to see that $r'_m = 1$ is possible for $m \leq m^*$ but causes no change in the proof other than one row being deleted in the corresponding table(s).

- $\mathbf{p}_{m^*} = \left(\begin{array}{c|cccc} Rank & p_m^1 & p_m^2 & p_m^3 & p_m^4 \\ \hline 1 & \cdot & \cdot & \cdot & a_{m^*} \\ r'_{m^*} & \cdot & \cdot & a_{m^*} & \cdot \\ r'_{m^*}+1 & a_{m^*} & \cdot & \cdot & \cdot \\ r_{m^*}+1 & \cdot & a_{m^*} & \cdot & \cdot \end{array}\right)$,

- $\forall m > m^*$, $\mathbf{p}_m = \left(\begin{array}{c|cccc} Rank & p_m^1 & p_m^2 & p_m^3 & p_m^4 \\ \hline 1 & b_m & a_m & a_m & a_m \\ 2 & \cdot & b_m & b_m & b_m \\ A_m & a_m & \cdot & \cdot & \cdot \end{array}\right)$

Let $\overrightarrow{a} = (a_1, \ldots, a_M)$, $\overrightarrow{b} = (b_1, \ldots, b_M)$, $\overrightarrow{c} = (a_{\leq m^*}, \overrightarrow{b}_{>m^*}) \in \mathcal{A}$, $\mathcal{A}' = \{(\overrightarrow{\alpha}_{\leq m^*}, b_{>m^*}) : \overrightarrow{\alpha} \in \mathcal{A}\} \setminus \{\overrightarrow{c}\}$. Now, extend $\mathbf{p}$ to coordinate profile $\overline{\mathbf{p}}$ involving $I \geq \prod_{m \leq m^*} A_m$ +3 voters and such that:

(1) $\forall i \in \{1,2,3\}$, $\overline{p}^i = p^i$,

(2) $\forall i \in \{4, \ldots, I\}$, $\overline{p}^i$ agrees with p^4 on the ranks given to a_m for all $m \in \mathcal{M}$, $\overline{p}^i$ agrees with p^4 on the ranks given to b_m for all $m > m^*$, and for all $m > m^*$, $\forall c_m \in \mathcal{A}_m \setminus \{a_m, b_m\}$ there exists $i(c_m) \in \{4, \ldots, I\}$ such that $r[c_m, \overline{p}^{i(c_m)}] = A_m$,

(3) $\forall \overrightarrow{d} \in \mathcal{A}'$, there exists $i(\overrightarrow{d}) \in \{4, \ldots, I\}$ such that $R[\overrightarrow{d}, \overline{p}^{i(\overrightarrow{d})}] = (R_{<m^*}, r_{m^*}+1, 2, \ldots, 2)$.

As $I^* \geq \prod_{m \leq m^*} A_m$, condition (3) can be satisfied. We have $\prod_{m \in \mathcal{M}} F_I(\overline{\mathbf{p}}) \subseteq \mathcal{A}' \cup \{\overrightarrow{c}\}$ and $\overrightarrow{c} \in \prod_{m \in \mathcal{M}} F_I(\overline{\mathbf{p}})$. Table 2 gives the rank vectors of $\overrightarrow{a}$ and $\overrightarrow{c}$ for each coordinate preference $\overline{p}^i$:

Table 2

i	$R(\overrightarrow{a}, \overline{p}^i)$	$R[\overrightarrow{c} = (a_{\leq m^*}, \overrightarrow{b}_{>m^*}), \overline{p}^i]$
1	$\tilde{R}' = (R_{<m^*}, r'_{m^*}+1, A_{m^*+1}, \ldots, A_M)$	$(R_{<m^*}, r'_{m^*}+1, 1, \ldots, 1)$
2	$\tilde{R} = (R_{<m^*}, r_{m^*}+1, 1, \ldots, 1)$	$(R_{<m^*}, r_{m^*}+1, 2, \ldots, 2)$
3	$(R_{<m^*}, r'_{m^*}, 1, \ldots, 1)$	$(R_{<m^*}, r'_{m^*}, 2, \ldots, 2)$
4	$\mathbf{1}$	$(\overbrace{1, \ldots, 1}^{m^*}, \overbrace{2, \ldots, 2}^{M-m^*})$

By the responsiveness of δ, $R(\overrightarrow{c}, \overline{p}^i)\ \delta\ R(\overrightarrow{c}, \overline{p}^2)$ for all $i \neq 2$. Thus, $V_I[\overrightarrow{c}, \boldsymbol{\delta}(\overline{\mathbf{p}})] = r[\overrightarrow{c}, \delta(\overline{p}^2)]$. By assumption, $R(\overrightarrow{a}, \overline{p}^1)\ \delta\ R(\overrightarrow{a}, \overline{p}^2)$, and by the responsiveness of δ, $R(\overrightarrow{a}, \overline{p}^i)\ \delta\ R(\overrightarrow{a}, \overline{p}^2)$ for all $i \geq 3$. Thus, $V_I[\overrightarrow{a}, \boldsymbol{\delta}(\overline{\mathbf{p}})] = r[\overrightarrow{a}, \delta(\overline{p}^2)]$. Moreover, again by the responsiveness of δ, $R(\overrightarrow{a}, \overline{p}^2)\ \delta\ R(\overrightarrow{c}, \overline{p}^2)$, which implies $V_I[\overrightarrow{a}, \boldsymbol{\delta}(\overline{\mathbf{p}})] < V_I[\overrightarrow{c}, \boldsymbol{\delta}(\overline{\mathbf{p}})]$. It follows that $\overrightarrow{c} \notin F_I[\boldsymbol{\delta}(\overline{\mathbf{p}})]$.

Finally, pick any $\overrightarrow{d} \in \mathcal{A}'$. By definition of $\overline{\mathbf{p}}$, $V_I[\overrightarrow{d}, \boldsymbol{\delta}(\overline{\mathbf{p}})] \geq r[\overrightarrow{d}, \delta(\overline{p}^{i(\overrightarrow{d})})] = r[\overrightarrow{c}, \delta(\overline{p}^2)] = V_I[\overrightarrow{c}, \boldsymbol{\delta}(\overline{\mathbf{p}})]$. As $V_I[\overrightarrow{a}, \boldsymbol{\delta}(\overline{\mathbf{p}})] < V_I[\overrightarrow{c}, \boldsymbol{\delta}(\overline{\mathbf{p}})]$, we get $[\prod_{m \in \mathcal{M}} F_I(\mathbf{p})] \cap F_I[\boldsymbol{\delta}(\overline{\mathbf{p}})] = \emptyset$, in contradiction the assumption that F_I is weakly stable over $\mathcal{D}$.

This shows that case 1 is impossible.

Case 2: $r_{m^*} = A_{m^*}$.

First, observe that one must have $r'_{m^*} = A_{m^*} - 1$. To see why, suppose that $r'_{m^*} < A_{m^*} - 1$. Define $R'' = (R_{<m^*}, A_{m^*} - 1, 1, \ldots, 1)$. By the responsiveness of δ, $R''\ \delta\ R$, and by the transitivity of δ, $R''\ \delta\ R'$. Since $r''_{m^*} = A_{m^*} - 1 < A_{m^*}$, this contradicts with case 1 being impossible. We complete the proof by using an argument similar to the one for case 1. Consider the coordinate 3-voter profile $\mathbf{p}$ with the form below:

- $\forall m < m^*$, $\mathbf{p}_m = \left(\begin{array}{c|ccc} Rank & p_m^1 & p_m^2 & p_m^3 \\ 1 & \cdot & \cdot & a_m \\ r_m & a_m & a_m & \cdot \end{array}\right)$,

- $\mathbf{p}_{m^*} = \left(\begin{array}{c|ccc} Rank & p^1_{m^*} & p^2_{m^*} & p^3_{m^*} \\ \hline 1 & \cdot & a_{m^*} & a_{m^*} \\ A_{m^*}-1 & b_{m^*} & b_{m^*} & b_{m^*} \\ A_{m^*} & a_{m^*} & \cdot & \cdot \end{array}\right)$,

- $\forall m > m^*$, $\mathbf{p}_m = \left(\begin{array}{c|ccc} Rank & 1 & 2 & 3 \\ \hline 1 & a_m & \cdot & a_m \\ A_m & \cdot & a_m & \cdot \end{array}\right)$.

Let $\overrightarrow{a} = (a_1, \dots, a_M)$, $\overrightarrow{b} = (b_1, \dots, b_M)$, $\overrightarrow{c} = (b_{m^*}, \overrightarrow{a}_{-m^*}) \in \mathcal{A}$, $\mathcal{A}' = \{(b_{m^*}, \overrightarrow{\alpha}_{-m^*}) : \overrightarrow{\alpha} \in \mathcal{A}\} \setminus \{\overrightarrow{c}\}$. Now, extend $\mathbf{p}$ to a coordinate profile $\overline{\mathbf{p}}$ involving $I \geq \prod_{m \leq m^*} A_m + 3$ and such that

(1) $\forall i \in \{1,2\}$, $\overline{p}^i = p^i$,

(2) $\forall i \in \{3,...,I\}$, $\overline{p}^i$ agrees with p^3 on the ranks given to a_m for all $m \in \mathcal{M}$, and $\overline{p}^i$ agrees with p^3 on the rank given to b_{m^*},and $\forall c_{m^*} \in \mathcal{A}_{m^*} \setminus \{a_{m^*}, b_{m^*}\}$ there exists $i(c_{m^*}) \in \{4, \dots, I\}$ such that $r[c_{m^*}, \overline{p}^{i(c_{m^*})}] = A_{m^*}$,

(3) $\forall \overrightarrow{d} \in \mathcal{A}'$, there exists $i(\overrightarrow{d}) \in \{3, \dots, I\}$ with $r[(\overrightarrow{d}, \overline{p}^{i(\overrightarrow{d})}] = R' = (R_{<m^*}, A_{m^*}-1, A_{m^*+1}, \dots, A_M)$.

As $I^* \geq \prod_{m \leq m^*} A_m$, condition (3) can be satisfied. Check that $\prod_{m \in \mathcal{M}} F_I(\overline{\mathbf{p}}_m) \subseteq \mathcal{A}' \cup \{\overrightarrow{c}\}$, and $\overrightarrow{c} \in \prod_{m \in \mathcal{M}} F_I(\overline{\mathbf{p}}_m)$. Table 3 gives the rank vectors of $\overrightarrow{a}$ and $\overrightarrow{c}$ for each $\overline{p}^i$:

Table 3

i	$R(\overrightarrow{a}, \overline{p}^i)$	$R[\overrightarrow{c} = (b_{m^*}, \overrightarrow{a}_{-m^*}), \overline{p}^i]$
1	$R = (R_{<m^*}, A_{m^*}, 1, \dots, 1)$	$(R_{<m^*}, A_{m^*}-1, 1, \dots, 1)$
2	$(R_{<m^*}, 1, A_{m^*+1}, \dots, A_M)$	$R' = (R_{<m^*}, A_{m^*}-1, A_{m^*+1}, \dots, A_M)$
$i \geq 3$	$\overrightarrow{1}$	$(\overbrace{1, \dots, 1}^{m^*-1}, A_{m^*}-1, \overbrace{1, \dots, 1}^{M-m^*})$

The responsiveness of δ implies $V_I[\overrightarrow{c}, \boldsymbol{\delta}(\overline{\mathbf{p}})] = r[\overrightarrow{c}, \delta(\overline{p}^2)]$. Moreover, again by the responsiveness of δ, we have $V_I[\overrightarrow{a}, \boldsymbol{\delta}(\overline{\mathbf{p}})] \in \{r[\overrightarrow{a}, \delta(\overline{p}^1)], r[\overrightarrow{a}, \delta(\overline{p}^2)]\}$. Since $R \; \delta \; R'$, then $r[\overrightarrow{a}, \delta(\overline{p}^1)] < V_I[\overrightarrow{c}, \boldsymbol{\delta}(\overline{\mathbf{p}})]$. Finally, the responsiveness of δ implies $r[\overrightarrow{a}, \delta(\overline{p}^2)] < V_I[\overrightarrow{c}, \boldsymbol{\delta}(\overline{\mathbf{p}})]$. Hence, $V_I[\overrightarrow{a}, \boldsymbol{\delta}(\overline{\mathbf{p}})] < V_I[\overrightarrow{c}, \boldsymbol{\delta}(\overline{\mathbf{p}})]$. Thus, $\overrightarrow{c} \notin F_I[\boldsymbol{\delta}(\overline{\mathbf{p}})]$.

Finally, pick any $\overrightarrow{d} \in \mathcal{A}'$. By definition of $\overline{\mathbf{p}}$, $V_I[\overrightarrow{d}, \boldsymbol{\delta}(\overline{\mathbf{p}})] \geq r[\overrightarrow{d}, \overline{p}^{i(\overrightarrow{d})}] = r[\overrightarrow{c}, \delta(\overline{p}^2)] = V_I[\overrightarrow{c}, \boldsymbol{\delta}(\overline{\mathbf{p}})]$. As $V_I[\overrightarrow{a}, \boldsymbol{\delta}(\overline{\mathbf{p}})] < V_I[\overrightarrow{c}, \boldsymbol{\delta}(\overline{\mathbf{p}})]$, we get $[\prod_{m \in \mathcal{M}} F_I(\mathbf{p})] \cap F_I[\boldsymbol{\delta}(\overline{\mathbf{p}})] = \emptyset$, in contradiction the assumption that F_I is weakly stable over $\mathcal{D}$.

The second step in the proof of the necessary part consists in showing that F_I is weak product stable only if all voters are assigned to the same lexicographic preference.

Lemma 7 *Take $\widetilde{q}, \widetilde{r} \in \mathcal{L}_{\mathcal{M}}$ with $\widetilde{q} \neq \widetilde{r}$. If $I \geq 3$, and if $\{\delta^{\widetilde{q}}, \delta^{\widetilde{r}}\} \subseteq \mathcal{D}$, then F_I is not weakly product stable over $\mathcal{D}$.*

Proof Take $\widetilde{q}, \widetilde{r} \in \mathcal{L}_{\mathcal{M}}$ with $\widetilde{q} \neq \widetilde{r}$ and $\{\delta^{\widetilde{q}}, \delta^{\widetilde{r}}\} \subseteq \mathcal{D}$. Up to a relabelling of coordinates $m \in \mathcal{M}$, one can suppose w.l.o.g. that $\widetilde{q} = [12 \dots M]$ and $m^* \; \widetilde{r} \; \overline{m}$ for some $\overline{m}, m^* \in \mathcal{M}$ with $\overline{m} < m^*$. Take a 3-voter coordinate profile $\mathbf{p}$ such as below:

- $\forall m \in \mathcal{M} \setminus \{m^*, \overline{m}\}$, $\mathbf{p}_m = \left(\begin{array}{c|ccc} Rank & p^1_m & p^2_m & p^3_m \\ \hline 1 & b_m & b_m & b_m \\ \dots & \dots & \dots & \dots \end{array}\right)$,

- $\mathbf{p}_{m^*} = \left(\begin{array}{c|ccc} Rank & p^1_{m^*} & p^2_{m^*} & p^3_{m^*} \\ \hline 1 & b_{m^*} & a_{m^*} & b_{m^*} \\ 2 & a_{m^*} & b_{m^*} & c_{m^*} \\ \dots & \dots & \dots & \dots \end{array}\right)$, $\mathbf{p}_{\overline{m}} = \left(\begin{array}{c|ccc} Rank & p^1_{\overline{m}} & p^2_{\overline{m}} & p^3_{\overline{m}} \\ \hline 1 & a_{\overline{m}} & b_{\overline{m}} & a_{\overline{m}} \\ 2 & b_{\overline{m}} & c_{\overline{m}} & b_{\overline{m}} \\ \dots & \dots & \dots & \dots \end{array}\right)$.

Consider $\overrightarrow{b} = (b_1, \ldots, b_M) \in \mathcal{A}$. Clearly, $\prod_{m \in \mathcal{M}} F_I(\mathbf{p}) = \{\overrightarrow{b}\}$. Now, define $\overrightarrow{c} = (c_1, \ldots, c_M) \in \mathcal{A}$ by $\forall m \in \mathcal{M}$, $c_m = \begin{cases} a_m & \text{if } m \in \{m^*, \overline{m}\} \\ b_m & \text{otherwise} \end{cases}$. Take $\boldsymbol{\delta} = (\delta^1, \delta^2, \delta^3) = (\delta^{\widetilde{q}}, \delta^{\widetilde{r}}, \delta^{\widetilde{q}})$. Since $a_{\overline{m}}\ p^i_{\overline{m}}\ b_{\overline{m}}$ for $i = 1, 3$, the definition of $\delta^{\widetilde{q}}$ ensures that $\overrightarrow{c}\ \delta(p^i)\ \overrightarrow{b}$ for $i = 1, 3$. Similarly, since $a_{m^*}\ p^2_{m^*}\ b_{m^*}$, the definition of $\delta^{\widetilde{r}}$ ensures that $\overrightarrow{c}\ \delta(p^2)\ \overrightarrow{b}$. Thus, $V_I[\overrightarrow{c}, \boldsymbol{\delta}(\mathbf{p})] < V_I[\overrightarrow{b}, \boldsymbol{\delta}(\mathbf{p})]$, which in turn implies $[\prod_{m \in \mathcal{M}} F_I(\mathbf{p})] \cap F_I[\boldsymbol{\delta}(\overline{\mathbf{p}})] = \emptyset$. This shows that F_I is not weakly product stable over $\mathcal{D}$.

The necessary part of theorem 1 follows from combining lemma 6 and lemma 7.